Low rank matrix recovery from few orthonormal basis measurements

*(Invited Paper)*

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**Abstract**—Recent insights concerning the PhaseLift algorithm for retrieving phases have furthered our understanding of low rank matrix recovery from rank-one projective measurements. Motivated by the structure of certain quantum mechanical experiments, we introduce a particular class of such rank-one measurements: orthonormal basis measurements. One such measurement corresponds to choosing an orthonormal basis and treating all the rank-one projectors onto different basis elements as a series of consecutive measurement matrices. We elaborate on performing low-rank matrix recovery from few, sufficiently random orthonormal basis measurements and sketch applications of such a procedure in quantum physics. We conclude this article by presenting numerical experiments testing such an approach.

**Index Terms**—Low rank matrix recovery, quantum information theory, phase retrieval

I. INTRODUCTION

A. Low rank matrix recovery

The young but already extensive field of low rank matrix recovery uses ideas from compressed sensing to reconstruct a given matrix of low rank from highly incomplete data in a computationally efficient way. Here we shall restrict our attention to hermitian $n \times n$ matrices which form a real $n^2$-dimensional vector space $H^n$. Let $X \in H^n$ be a rank-$r$ matrix of interest ($r \ll n$) and suppose that we have access to $m$ linear measurements of the form

$$y_i = \text{tr}(A_i X) \quad i = 1, \ldots, m,$$

(1)

where $A_1, \ldots, A_m \in H^n$ denote measurement matrices. Having data of this form at hand, the analogy to compressed sensing [1] suggests to exploit the low-rank structure of $X$ by means of a constrained nuclear-norm minimization

$$\min_{Z \in H^n} \|Z\|_* \quad \text{subject to} \quad \text{tr}(ZA_i) = y_i \quad i = 1, \ldots, m,$$

(2)

which can be solved computationally efficiently. One aim of low-rank matrix recovery is to identify instances for which $m = Cn \log(n)$ measurements of the form (1) suffice to prove that the convex program (2) recovers the sought for $X$ with high probability. Up to date many such instances have been identified [2]–[8].

B. The phase retrieval problem

The problem of retrieving a complex signal $x \in \mathbb{C}^n$ from measurements of the form

$$y_i = |\langle a_i, x \rangle|^2 \quad i = 1, \ldots, m,$$

(3)

where $a_1, \ldots, a_m \in \mathbb{C}^n$ are measurement vectors, has long been abundant in many areas of science. Recently, its mathematical structure has received considerable attention in its own right. The problem is clearly ill-posed, since all phase information is lost in the measurement process and the measurements (3) are furthermore of a non-linear nature. This second obstacle can be overcome by a trick [9] well known in conic programming: the quadratic expressions (3) are linear in the outer products $xx^*$ and $a_i a_i^*$:

$$y_i = |\langle a_i, x \rangle|^2 = \text{tr}((a_i a_i^*)(xx^*)).$$

Since the object of interest – $X := xx^* \in H^n$ – is proportional to a rank-one projector, such a “lift” turns the phase retrieval problem into a particular instance of low rank matrix recovery – a fact that was first observed by Candès, Eldar, Strohmer and Voroninski [10]. In turn, the measurement matrices $A_i = a_i a_i^*$ are constrained to be proportional to rank-one projectors as well. These structural constraints prevent a direct application of results from low-rank matrix recovery, because signal and measurements fail to be sufficiently incoherent$^2$ in the sense of [5], [11]. Nonetheless, phase retrieval recovery guarantees by means of the optimization (2) – dubbed PhaseLift for this particular setting – have been established for different types of measurements.

The chronologically first result [12] of this kind proves a non-uniform recovery guarantee for $m = Cn \log(n)$ measurement vectors $a_i$ sampled independently and uniformly from the complex unit sphere. This recovery guarantee was partially derandomized (at the cost of a multiplicative constant) for measurement vectors drawn independently and uniformly from the complex unit sphere.

In a sense, the nuclear norm $\|X\|_* = \text{tr}(|X|)$ is the natural non-commutative analogue of the $\ell_1$-norm which features prominently in compressed sensing [1]. Furthermore, low rank assures that the matrix of interest is sparse in its eigenbasis.

$^1$In fact, both references establish an optimal sampling rate (up to a multiplicative constant) for measurement vectors drawn independently and uniformly from the complex unit sphere.

$^2$Roughly, the incoherence parameter captures the well-posedness of the inverse problem.
that are randomly distorted. To be more concrete, one such measurement encompasses \( n \) distorted Fourier vectors of the form \( a_{k l} = D_l f_k \) \((1 \leq i \leq k)\), where each \( D_l \) \((1 \leq l \leq L)\) is an instance of a random matrix diagonal in the standard basis – e.g. a diagonally embedded Rademacher vector with random erasures. Subsequently, a recovery guarantee requiring fewer – namely \( L = C \log^2(n) \) – such coded diffraction patterns was established in [17].

We conclude this section by pointing out that PhaseLift is just one possibility for solving the phase retrieval problem. Other approaches rely on polarization identities [18], alternate projections [19], or Wirtinger flow methods [20].

II. LOW RANK MATRIX RECOVERY FROM ORTHONORMAL BASIS MEASUREMENTS

Given these recent advances regarding the phase retrieval problem, it seems natural to ask, whether these insights can be translated to general low rank matrix recovery from certain types of rank-one projective measurements. A first step in this direction was done in [15], where uniform recovery guarantees for inferring hermitian rank \( r \)-matrices from \( m = C r n \) projectors onto i.i.d Gauss-random vectors, or from \( m = C r n \log(n) \) projectors onto randomly chosen elements of a spherical 4-design, were established.

Here – inspired by coded diffraction patterns [16], [17] – we shall focus on randomly distorted basis measurements instead. More formally: let \( X \in \mathbb{H}^{n} \) be a rank-\( r \) matrix of interest and consider \( n \) consecutive measurements of the form

\[
y_{1,l} = \text{tr}((D_l b_l b_l^* D_l^*) X) = \langle D_l b_l, X D_l b_l \rangle,
\]

\[
\vdots
\]

\[
y_{n,l} = \text{tr}((D_l b_n b_n^* D_l^*) X) = \langle D_l b_n, X D_l b_n \rangle,
\]

where \( b_1, \ldots, b_n \) denotes an arbitrary orthonormal basis of \( \mathbb{C}^n \) and \( D_l \) is an instance of a random \( n \times n \) matrix. Motivated by typical quantum mechanical experiments – see Section III – we consider the special case, where each \( D_l \) is unitary. Consequently each distorted orthonormal basis measurement corresponds to measuring a different orthonormal basis \( b_1^{(l)}, \ldots, b_n^{(l)} \):

\[
y_{1,l} = \text{tr}(b_1^{(l)} (b_1^{(l)})^* X) = \langle b_1^{(l)}, X b_1^{(l)} \rangle,
\]

\[
\vdots
\]

\[
y_{n,l} = \text{tr}(b_n^{(l)} (b_n^{(l)})^* X) = \langle b_n^{(l)}, X b_n^{(l)} \rangle.
\]

Regarding such types of measurements, the following question is imminent:

Are there unitary transformations \( D_l \) – or equivalently: orthonormal bases \( b_1^{(l)}, \ldots, b_n^{(l)} \) – such that the convex optimization (2) allows for recovering an unknown rank-\( r \) matrix \( X \) from \( L = C r \text{polylog}(n) \) orthonormal basis measurements of the form (4), or (5), respectively?

It is highly conceivable, that this is the case for unitaries \( D_l \) chosen uniformly from the Haar measure – or equivalently: orthonormal bases \( b_1^{(l)}, \ldots, b_n^{(l)} \) obtained by choosing \( n \) standard complex Gaussian vectors independently at random and orthonormalizing them (e.g. by means of Gram-Schmidt). Clearly, such a procedure requires one to be able to choose from a continuous, very generic union of bases. However, the results in [13], [15] suggest that such a requirement might not be necessary and that more structured, finite unions of bases may suffice to establish low rank matrix recovery guarantees by means of nuclear norm minimization. For going further into that direction – and, by doing so, partially derandomizing the recovery scheme proposed above – we rely on the concept of spherical designs. These finite sets of unit vectors were first introduced in [21] and serve as a general purpose tool for partial derandomization – see [13], [22] for further reading on this aspect of spherical designs. To mimic the problem’s structure, we need to equip spherical designs with an additional structural property. This results in the following definition.

**Definition 1** (spherical \( t \)-design with basis structure). We call a finite union \( \Lambda_t = \{b_1^{(i)}, \ldots, b_n^{(i)}\}_{i=1}^N \subset \mathbb{C}^n \) of orthonormal bases a spherical \( t \)-design with basis structure, if the uniform distribution over the \( N n \) elements of \( \Lambda_t \) reproduces the first \( 2t \) moments of standard complex Gaussian vectors renormalized to unit length.

Although demanding such an orthonormal basis structure in addition to the \( t \)-design property might seem alien at first sight, there are numerous examples for designs that admit it. Examples include arbitrary orthonormal bases (1-designs), maximal sets of mutually unbiased bases (2-designs) [23] which exist in prime power dimensions, stabilizer states (3-designs in power-of-two-dimensions) [24] and orbits \( \{U b_1, \ldots, U b_n\}_{U=1}^N \) of an arbitrary orthonormal basis under the action of a unitary \( t \)-design \( \{U_i\}_{i=1}^N \) [25], [26] (which constitute a spherical \( t \)-designs of the same order).

Similar to [13], \( t \)-designs with basis structure suffice to establish a non-uniform recovery guarantee for measurements of the form (5):

**Theorem 2** (Low rank matrix recovery from orthonormal basis measurements). Let \( X \in \mathbb{H}^{n} \) be an arbitrary matrix of rank \( r \) and let \( \Lambda_t \) be a \( t \)-design (\( t \geq 3 \)) with basis structure in the sense of Definition 1. Then choosing

\[
L = C t n^{2/t} \log^2(n)
\]

different bases independently and uniformly at random from \( \Lambda_t \) and performing the corresponding orthonormal basis measurements of the form (5) suffices to recover \( X \) by means of the convex optimization (2) with high probability.

Note that the requirement \( t \geq 3 \) on the design order is in fact necessary. Similar to [13, Theorem 2], a counter-example can be constructed for \( t = 2 \) using mutually unbiased bases. While Theorem 2 is non-trivial – as it allows for recovering \( X \) from a total of \( m = L n = C t n^{1+2/t} \log^2(n) \ll n^2 \) measurements (provided that \( r \ll n \) is large enough) – the required number \( L \) of orthonormal basis measurements contains the term \( n^{2/t} \). As a consequence, the sampling rate only becomes optimal up to polylog-factors, if we allow the design order \( t \) to grow logarithmically with the dimension (\( t = 2 \log(n) \)). However, we believe that the factor \( n^{2/t} \) in (6) is an artifact of the proof technique employed. It uses ideas presented in [13] which resulted in a similar non-optimal factor appearing in the sampling rate. There, employing different techniques allowed for eradicating such a factor and substantially strengthening the statement [15]. In turn, we believe that a more careful analysis will allow for

\(^4\)Unitary \( t \)-design are a natural generalization of the spherical design concept to unitaries. These finite sets of unitary matrices reproduce the Haar-uniform distribution over the unitary group up to 2\( t \)-th moments.
establishing a recovery guarantee getting by with a sampling rate that already for \( t = 3 \), or \( t = 4 \), is optimal up to polylog-factors.

Furthermore, we want to point out that there is crucial difference between Theorem 2 and the main result established in [13] (and its generalization presented in [15]). There it is assumed that each measurement is sampled independently and uniformly from \( \Lambda_t \) – which strongly resembles the design’s defining property (see Definition 1). In Theorem 2, on the other hand, one entire basis is selected at random and \( n \) corresponding orthonormal basis measurements of the form (5) are carried out. Evidently, these \( n \) measurements are correlated. Such a situation bears more similarity to coded diffraction patterns [16], [17] than it does to independent sampling of individual design elements. In order to establish Theorem 2 we pay tribute to this fact and combine proof techniques from [17] (which can handle such correlated measurements) with others from [13] (that exploit the underlying design-structure). A detailed presentation of such a proof would go beyond the scope of this article and will be presented elsewhere.

Finally we want to point out that Theorem 2 is stated for noiseless measurements only. We leave establishing stability towards noise for future work.

III. MOTIVATION: QUANTUM STATE TOMOGRAPHY

In this section, we briefly want to motivate the measurement setups introduced in (4) and (5) without going into too much detail. For further reading on the topics introduced here, we defer the interested reader to [27, Chapter 2.2]. In quantum mechanics, the state of an isolated finite \( n \)-dimensional quantum system is fully described by a positive-semidefinite hermitian matrix \( \rho \in H^n \) with unit trace. Such a matrix is called a density operator. Estimating the density operator of a given quantum system is an important task in quantum physics known as quantum state tomography [28], [29]. When performing this task, it is highly desirable to exploit additional structure – if present – especially when \( n \) is large\(^5\). One such structural property – often encountered in actual experiments – is approximate purity, i.e. the density operator \( \rho \) is well approximated by a low rank matrix. Performing quantum state tomography under the prior assumption of approximate purity therefore constitutes a particular instance of low rank matrix recovery [31], [32].

The dynamics of an isolated quantum system – i.e. some physical evolution – corresponds to a unitary transformation \( \rho \mapsto \rho' = U\rho U^* \) of the system’s density operator \( \rho \).

Finally, after preparing a quantum system \( \rho \) and letting it undergo some physical evolution \( U \), a typical experiment is terminated by performing a measurement on the resulting system \( \rho' \). While substantially more general types of measurements are possible, non-degenerate projective measurements constitute a particular important subclass. Each such measurement is described by a non-degenerate hermitian matrix \( M = \sum_{i=1}^n \lambda_i b_i b_i^* \) with eigenvalues \( \lambda_i \in \mathbb{R} \) and a corresponding orthonormal eigenbasis \( \{b_1, \ldots, b_n\} \subset \mathbb{C}^n \). Upon performing such a measurement on a system described by \( \rho \), quantum mechanics postulates that the probability of obtaining the outcome \( \lambda_i \) is given by

\[
p(\lambda_i, \rho) = \text{tr} \left( b_i^* b_i \rho \right) = \langle b_i, \rho b_i \rangle.
\]

Repeating such an experiment (i.e. preparing \( \rho \) and measuring \( M \)) many times allows one to estimate the \( n \) probabilities \( p(\lambda_i, \rho) \) even more accurately.

Consequently, combining a certain unitary evolution \( U \) of a density operator \( \rho \) with performing a non-degenerate projective measurement \( M = \sum_{i=1}^n \lambda_i b_i b_i^* \), results in estimating \( n \) numbers of the form

\[
p(\lambda_i, \rho) = \text{tr} \left( b_i^* b_i U X U^* \right) = \langle U^* b_i, X U^* b_i \rangle \quad i = 1, \ldots, n. \quad (7)
\]

This exactly corresponds to one orthonormal basis measurement introduced in (4) (with \( D_1 = U^* \)) and the corresponding experiment is sketched in Figure 1. With such a setup at hand, Theorem 2 yields the following corollary relevant for quantum state tomography.

![Fig. 1. Pictorial description of a typical quantum mechanical experiment. In a first step, a quantum system described by a density operator \( \rho \) is produced. The system then undergoes some physical evolution characterized by a unitary matrix \( U \). \rho \mapsto \rho' = U\rho U^* \). The experiment is concluded by performing a measurement \( M \). If \( M = \sum_{i=1}^n \lambda_i b_i b_i^* \) is a non-degenerate projective measurement, information about \( \rho' \) can be gained via (7) by repeating the experiment many times and inferring the probabilities \( p(\lambda_i, \rho') \) of the individual measurement outcomes \( \lambda_i \) occurring.](image-url)

Corollary 3 (Quantum state tomography from sufficiently random evolutions). Let \( \rho \) be a density operator of rank \( r \leq n \) and let \( M = \sum_{i=1}^n \lambda_i b_i b_i^* \) denote a fixed non-degenerate projective measurement. Then, \( L = C r \log(n) \) independent instances of the basic experimental protocol described in Figure 1 suffice to recover \( \rho \) via (2) with high probability, provided that the unitary evolutions are chosen from a sufficiently generic set – e.g. a unitary \( 2\log(n) \)-design.

Some remarks on the practicality of the protocol presented in Corollary 3 may be appropriate: The postulates of quantum mechanics demand that each instance of the scenario depicted in Figure 1 needs to be repeated many times in order to infer the resulting probability distribution. This obstacle is of a fundamental nature and cannot easily be overcome. However, when it comes to imposing evolutions, some unitaries are considerably more challenging to realize than others. While the effort for implementing a generic Haar-random unitary evolution is considerable, implementing an evolution corresponding to a random element of a weighted, approximate unitary \( t \)-design can be done much more easily [33]. Practicality issues of this type were our main motivation for focusing on \( t \)-designs with basis structure, as they include orbits \( \{U b_1, \ldots, U b_n\} \) of any orthonormal basis under a a the action of a unitary \( t \)-design as a special case. Consequently, the results in [33] assure that the \( L \) different instances of the experiment proposed in Corollary 3 can be implemented in a practical way\(^6\).

Note that Corollary 3 is not the first approach to use low rank matrix recovery techniques for quantum state tomography. Up to now, recovery of approximately pure density operators by means of the convex optimization problem (2) has been established for

\(^5\)Technically, this conclusion is only valid if Theorem 2 remains true for weighted, approximate \( t \)-designs with basis structure. That this is indeed the case, will be established elsewhere.
independently chosen (generalized) Pauli measurements [5], [31] which can be implemented in a practical way for various experimental setups. For this type of measurements, the statistics is well understood [32], uniform recovery guarantees have been established [6] and the procedure has been tested in experiments [34]. However, all the existing results manifestly require performing at least \( m = C r n \log(n) \) independently chosen Pauli-type measurements, each of which can be interpreted as a highly degenerate projective measurement\(^7\). Here, we propose and establish a novel approach that goes beyond the Pauli setting and exploits a much more fine-grained measurement outcome statistics. Arguably, our protocol requires a more complicated experimental setup and the theoretical assertions are weaker (so far), but it gets by with only \( L = C r \log^3(n) \) different measurement settings.

IV. Numerical Experiments

Finally, we complement our theoretical observations and claims with numerical experiments. These were implemented in Matlab, using CVX [35]. To this end, we used stabilizers states [27, Chapter 10.5] – a highly structured union of orthonormal bases that forms a 3-design in power-of-two-dimensions [24] (this is false for other dimensions). Due to their rich combinatorial structure, choosing one stabilizer basis independently at random can be implemented efficiently and we have used this in our numerical simulations. The results for dimensions \( n = 16 \) and \( n = 32 \) are depicted in Figure 2. In each case we ran a total of 30 independent experiments for matrix ranks between 1 and \( 3n/4 \) (\( x \)-axis) and the number \( L \) of measured stabilizer bases ranging from 1 to 70 and 1 to 120, respectively (\( y \)-axis). For each experiment we first constructed a rank-\( r \) test matrix \( X = \sum_{i=1}^{r} v_i v_i^* \), where each \( v_i \in \mathbb{C}^n \) was a standard Gaussian random vector and renormalized \( X \) to Frobenius norm one. We then chose \( L \) stabilizer bases uniformly at random and for each such basis, we evaluated the \( n \) measurement outcomes \( y_{1,1}, \ldots, y_{n,n} \) according to (5). Using these \( Ln \) data points, we ran the convex optimization (2) and declared the recovery a “success” if the Frobenius-norm distance between the reconstructed matrix \( X^2 \) and the true test signal \( X \) was smaller than \( 10^{-3} \). Figure 2 illustrates the resulting empirical success probability for dimensions \( n = 16 \) and \( n = 32 \): black corresponds to only failures, white to exclusively successes.

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\(^7\)In power of two dimensions, for instance, each non-trivial Pauli measurement has two eigenvalues ±1 with associated eigenspaces of dimension \( n/2 \) each.

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Fig. 2. Phase diagrams for rank-\( r \) matrix recovery via (2) from measuring random (orthonormal) stabilizer bases in dimensions \( n = 16 \) and \( n = 32 \). The \( x \)-axis specifies the rank of the test matrices – ranging from 1 to \( 3n/4 \) – while the \( y \)-axis denotes the number \( L \) of independently chosen stabilizer bases measured. The frequency of successful recovery via (2) over 30 independent runs is color-coded from black (zero) to white (one).
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