Abstract—We describe a fast method for the evaluation of an arbitrary high-dimensional multivariate algebraic polynomial in Chebyshev form at the nodes of an arbitrary rank-1 Chebyshev lattice. Our main focus is on conditions on rank-1 Chebyshev lattices allowing for the exact reconstruction of such polynomials from samples along such lattices. We present an algorithm for constructing suitable rank-1 Chebyshev lattices based on a component-by-component approach. Moreover, we give a method for the fast and exact reconstruction.

I. INTRODUCTION

We denote the Chebyshev polynomials of the first kind by $T_k : [−1, 1] → [−1, 1], T_k(x) := \cos(k \arccos x), k ∈ \mathbb{N}_0$. Note that for each $k ∈ \mathbb{N}_0$, $T_k$ is an algebraic polynomial of degree $\text{deg}(T_k) = k$ restricted to the domain $[−1, 1]$. Moreover, we define the multivariate Chebyshev polynomials $T_k : [−1, 1]^d → [−1, 1], T_k(x) := \prod_{i=1}^d T_{k_i}(x_i)$ for $d ∈ \mathbb{N}$, $x := (x_1, \ldots, x_d)\top ∈ [−1, 1]^d$ and $k := (k_1, \ldots, k_d)\top ∈ \mathbb{N}_0^d$. Let $\Pi_I := \text{span} \{ T_k(\hat{\omega}) : k ∈ I \}$, where $I ⊂ \mathbb{N}_0^d$, $d ∈ \mathbb{N}$, is a non-negative index set of finite cardinality, $|I| < \infty$. Then, each multivariate polynomial $p ∈ \Pi_I$ can be written as

$$p(x) = \sum_{k ∈ I} \hat{p}_k T_k(x) = \sum_{k ∈ I} \hat{p}_k \prod_{i=1}^d T_{k_i}(x_i), \quad \hat{p}_k ∈ \mathbb{R}, \quad (1)$$

where $x ∈ [−1, 1]^d$. We remark that if the index set $I = I_n^d := \{ k ∈ \mathbb{N}_0^d : \|k\|_1 ≤ n \}$, $n ∈ \mathbb{N}_0$, is the $\ell_1$-ball, then $\Pi_I$ is the space of all algebraic polynomials of (total) degree $≤ n$ in $d$ variables restricted to the domain $[−1, 1]^d$. Moreover, polynomials with hyperbolic cross index sets $I = H^d_n := \{ k ∈ \mathbb{N}_0^d : \prod_{i=1}^d \max(1, |k_i|) ≤ n \}$, where $n, d ∈ \mathbb{N}$, have already been used for approximations in sparse high-dimensional spectral Galerkin methods, cf. [1, Section 8.5].

In this paper, for a given arbitrary index set $I ⊂ \mathbb{N}_0^d$ of finite cardinality, we present a method for the fast evaluation of a polynomial $p$ from (1) at the nodes $x_j := \cos(\frac{2\pi j}{M} z), j = 0, \ldots, M$, of a $d$-dimensional rank-1 Chebyshev lattice

$$\text{CL}(z, M) := \{ x_j := \cos \left( \frac{2\pi j}{M} z \right) : j = 0, \ldots, M \}$$

with the generating vector $z ∈ \mathbb{N}_0^d$ and the size parameter $M ∈ \mathbb{N}_0$, where the cosine is applied component-wise. For a more general definition of $d$-dimensional rank-$k$ Chebyshev lattices, we refer to [2]. Moreover, we discuss conditions on a rank-1 Chebyshev lattice $\text{CL}(z, M)$ such that the fast and exact reconstruction of all coefficients $\hat{p}_k, k ∈ I$, from sampling values $p(x_j)$ taken at the corresponding nodes $x_j, j = 0, \ldots, M$, is possible. Both, for the fast evaluation and reconstruction, we only apply a single one-dimensional discrete cosine transform of type I (DCT-I) and additionally compute simple index transforms, see also [3]. Note that for the special case $I = I_n^d$, constructions of rank-1 Chebyshev lattices suitable for the exact reconstruction were already discussed in [2], [4] and the references therein. Here, we present an algorithm based on component-by-component (CBC) construction for arbitrary index sets $I ⊂ \mathbb{N}_0^d$ using ideas from [5]–[7].

We remark that our considerations for the reconstruction of the coefficients $\hat{p}_k, k ∈ I$, of a polynomial $p$ from (1) with known index set $I ⊂ \mathbb{N}_0^d$ in this paper establish a basis for the reconstruction of a polynomial $p$ with unknown index set $I$ using a method similar to the one presented in [8].

The remaining parts of this paper are organized as follows: In Section II, we give prerequisites for the subsequent sections. We discuss the fast evaluation and reconstruction in Section III. In Section IV, we point out relations of our results to existing work. Afterwards, in Section V, we present computed rank-1 Chebyshev lattices suitable for reconstruction. Finally, in Section VI, we summarize the results of this paper.

II. PREREQUISITES

A. One-dimensional DCT-I

First, we recall results for the fast reconstruction of a one-dimensional (algebraic) polynomial $p$. We are able to reconstruct the coefficients $\hat{p}_0, \ldots, \hat{p}_n ∈ \mathbb{R}$ of a polynomial $p$ from (1) with $I := I_n^1$ from sampling values $p(x_j)$ at the Chebyshev nodes $x_j := \cos(j\pi/n), j = 0, \ldots, n$. For this, we apply a one-dimensional DCT-I to the sampling values $p(x_j)$ and we obtain $\tilde{a}_k := \sum_{j=0}^n (\epsilon_j^2) p(x_j) \cos(jk\pi/n) =$
\[
\sum_{k' \in I_1'} \hat{p}_{k'} \sum_{n=0}^{N} (\xi_n^2) \cos(jk'\pi/n) \cos(jk\pi/n) \text{ for } k \in I_1',
\]

\[
\hat{M}(I_1') := \{h \in \mathbb{Z}^d : (|h_1|, \ldots, |h_d|)^\top \in I_1'\},
\]

which contains all frequencies \( k \in I \) and versions of these frequencies \( k \) mirrored at all coordinate axes. Moreover, we define the index sets

\[
\mathcal{M}_s(I) := \{h \in \mathcal{M}(I) : h_s \geq 0\}, s = \{1, \ldots, d\},
\]

which contain all frequencies \( k \in I \) and versions of these frequencies mirrored at all coordinate axes except the \( s \)-th. For instance, in the case \( d = 2 \) and \( n = 8 \), the index set \( I_8^2 \) as well as the corresponding extended symmetric index set \( \mathcal{M}(I_8^2) \) and mirrored index sets \( \mathcal{M}_1(I_8^2), \mathcal{M}_2(I_8^2) \) are depicted in Fig. 1.

Next, we remark that for \( y_1, y_2 \in \mathbb{R} \), we have

\[
\cos(y_1) \cos(y_2) = \frac{1}{2} (\cos(y_1 + y_2) + \cos(y_1 - y_2)).
\]

Using induction on the dimension \( d \in \mathbb{N} \) and due to \( \cos(x) = \cos(-x) \) for all \( x \in \mathbb{R} \), we obtain for \( y := (y_1, \ldots, y_d)^\top \in \mathbb{R}^d \)

\[
\prod_{t=1}^{d} \cos(y_t) = \sum_{m \in \mathcal{M}_s(I)} \frac{1}{2^{d-1}} \cos \left( \frac{m \cdot y}{2} \right),
\]

where \( \mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{N}^d \) and \( m \cdot y := \sum_{t=1}^{d} m_t y_t \).

III. FAST EVALUATION AND RECONSTRUCTION OF MULTIVARIATE POLYNOMIALS FROM \( \Pi_I \) ALONG RANK-1 CHEBYSHEV LATTICES USING DCT-I

A. Fast evaluation at the nodes of rank-1 Chebyshev lattices

Briefly, we describe a simple method for the fast evaluation of a polynomial \( p \) from (1) with arbitrary index set \( I \subset \mathbb{N}_0^d \)

\[
\hat{a}_t := \sum_{j=0}^{M} (\xi_j^M)^2 \hat{p}_j \cos \left( \frac{j}{M} \pi k_t z_t \right) = \sum_{j=0}^{M} (\xi_j^M)^2 \left( \prod_{l=1}^{d} \cos \left( \frac{j}{M} \pi k_l z_l \right) \right) \cos \left( \frac{j}{M} \pi \right)
\]

at the nodes \( x_j := \cos(\frac{j}{M} \pi z), j = 0, \ldots, M, \) of an arbitrary \( d \)-dimensional rank-1 Chebyshev lattice \( \text{CL}(z, M) \). Examples for two-dimensional rank-1 Chebyshev lattices are shown in Fig. 2. We remark that not all \( (M + 1) \) nodes \( x_j, j = 0, \ldots, M, \) have to be distinct, i.e., \( \text{CL}(z, M) \). see e.g. Fig. 2a. Due to (3), we have

\[
p(x_j) = \sum_{k \in I} \hat{p}_{k} \left( \prod_{l=1}^{d} \cos \left( \frac{jk_l \pi}{M} \right) \right) \cos \left( \frac{jk \pi}{M} \right).
\]

Therefore, for any \( s \in \{1, \ldots, d\} \), we build the index set \( \mathcal{M}_s(I) \) and we compute the coefficients \( \hat{b}_l \) by (5) for \( l \in I_1'M \). Then, we apply a one-dimensional DCT-I to these coefficients \( \hat{b}_l \) and this yields the function values \( p(x_j) \) for \( j = 0, \ldots, M \). In total, we require \( O(M \log M + d 2^d |I|) \) arithmetic operations.
for \( l \in I^1_M \). Due to (4), this means \( \hat{a}_l = \sum_{k \in I^1_M} \frac{\hat{p}_k}{2^{d-1}} M_m \sum_{M} \left( \sum_{j=0}^{M} (\cos(j \pi (m \circ k) \cdot z)) \cos(j \pi z) \right) \) for all \( m \in M(\{1\}) \). Since we have \( \{m \circ k : m \in M(\{1\})\} = M(\{k\}) \) for \( k \in I \) and due to the orthogonality condition (2), we are able to exactly reconstruct all the coefficients \( \hat{p}_k \) for \( k \in I \), of the polynomial \( p \) from (1) using the computed coefficients \( \hat{a}_l \). Let \( I \) be the index set \( I \subset \mathbb{N}_0^d \) and any \( M \in \mathbb{N}_0^d \) such that \( \|I\| = I_{\text{start}} \). If condition (9) is valid for \( I := (k_1, \ldots, k_t) \in I_{\text{input}} \) and \( M := (z_1, \ldots, z_t) \in \mathbb{Z}^t \), the index set \( I := I_{\text{input}} \), then \( \hat{p}_k = \frac{2^d}{|I|} \sum_{m \in M(\{1\})} (m \circ k) \cdot z_{\text{mod}} M = l \) with \( l := k \cdot z_{\text{mod}} M \) for all \( k \in I \) and \( M \in \mathbb{N}_0^d \) such that \( \|M\| = I_{\text{start}} \). Using a fast algorithm for the DCT-I, this computation can be performed in \( O(M \log M + d^2 |I|) \) arithmetic operations.

Again, we stress that the index set \( I \subset \mathbb{N}_0^d \), \( |I| < \infty \), may be arbitrarily chosen. Upper bounds on the size parameter \( M \) for the existence of a rank-1 Chebyshev lattice \( \mathbb{CL}(z, M) \) fulfilling condition (7) are discussed in Section IV-B. A method for the construction of a suitable generating vector \( z \in \mathbb{N}_0^d \) is described in the following subsection.

### Construction of suitable rank-1 Chebyshev lattices

In Fig. 5, we present an algorithm for the construction of a rank-1 Chebyshev lattice \( \mathbb{CL}(z, M) \) which allows for the exact reconstruction of the coefficients \( \hat{p}_k \), \( k \in I \), of a polynomial \( p \) from (1) based on samples taken at the nodes of
IV. RELATIONS TO EXISTING WORK

A. Padua points and higher-dimensional rank-s Chebyshev lattices

In [10], special sampling points were discussed in the two-dimensional case, so-called Padua points. For a parameter $a$ in $\mathbb{N}$, there are the nodes $x_j := (\cos(j\pi/(n+1)), \cos(j\pi/n))^\top$, $j = 0, \ldots, M$, of the rank-1 Chebyshev lattice $A_n := \text{CL}(z, M)$, where the generating vector $z := (n, n+1)^\top$ and the size parameter $M := n(n+1)$. As discussed in [10, Section 2], the Padua point set $A_n$ only consists of $n^2$ distinct points, whereas $M = n^2 + n$.

Lemma IV.1. Let the index set $I := I_n^2 := \{k \in \mathbb{N}_0^2 : k_1 + k_2 \leq n\}$, $n \in \mathbb{N}_0$, be the $\ell_1$-ball. Then, condition (7) is fulfilled and we can exactly reconstruct the coefficients $\hat{p}_j$, $k \in I$, of a polynomial $p$ from (1) at sampling values at the nodes of the Padua point set $A_n$ using (6).

Proof. The assertion follows from the Lagrange interpolation formula [11, (7c)]. Alternatively, condition (9) from Corollary III.3 can be verified.

In [4], an extensive search for higher-rank Chebyshev lattices allowing for the reconstruction of polynomials $p$ from (1) with $\ell_1$-ball index sets $I := I_n^d$ was performed and numerical results for the cases $d = 3, 4, 5$ were presented.

B. Reconstructing rank-1 lattices of multivariate trigonometric polynomials

In the following, we briefly show the relation to reconstructing rank-1 lattices of multivariate trigonometric polynomials from [7].

Theorem IV.2. Let $I \subset \mathbb{N}_0^d$ be an arbitrary index set of finite cardinality, $|I| < \infty$. Moreover, let $\Lambda(z, M) := \{y_j := \frac{1}{M}z \mod 1 : j = 0, \ldots, M - 1\}$ be a reconstructing rank-1 lattice with generating vector $z \in \mathbb{N}_0^d$ and even rank-1 lattice size $M \in 2\mathbb{N}$ for the extended symmetric index set $\mathcal{M}(I)$, i.e.,

$$h \cdot z \neq h' \cdot z \mod \hat{M} \text{ for all } h, h' \in \mathcal{M}(I), h \neq h'.$$

Then, the rank-1 Chebyshev lattice $\text{CL}(z, \frac{\hat{M}}{2})$ fulfills condition (7), i.e., we are able to exactly reconstruct the coefficients of a polynomial from (1) using samples at the nodes of $\text{CL}(z, \frac{\hat{M}}{2})$.

Proof. We consider the values

$$h \cdot z \mod \hat{M} = \begin{cases} h \cdot z \mod \hat{M}, & h \cdot z \mod \hat{M} \leq \frac{\hat{M}}{2}, \\ \hat{M} - (h \cdot z \mod \hat{M}), & \text{else} \end{cases}$$

for $h \in \mathcal{M}(I)$. Due to property (10), all values $h \cdot z$ mod $\hat{M}$ are distinct for $h \in \mathcal{M}(I)$ and we obtain for each $I \in I_{\hat{M}/2}$ that one of the following three cases may occur: Either

1. exactly two distinct frequencies $h, h' \in \mathcal{M}(I)$ exist such that $h \cdot z$ mod $\frac{\hat{M}}{2} = h' \cdot z$ mod $\frac{\hat{M}}{2}$, or
2. exactly one frequency $h \in \mathcal{M}(I)$ exists such that $h \cdot z$ mod $\frac{\hat{M}}{2} = l$, or
3. such a frequency does not exist for $l$.

In the first case, $h' = -h$ follows, since for each $h \in \mathcal{M}(I) \setminus \{0\}$, also the frequency $-h \in \mathcal{M}(I) \setminus \{0\}$ and we have (8) with $\hat{M} := \frac{\hat{M}}{2}$, i.e., $(-h) \cdot z$ mod $\frac{\hat{M}}{2} = h \cdot z$ mod $\frac{\hat{M}}{2} = l$. The second case can only occur for $h = 0$, since otherwise the (non-zero) frequency $-h \in \mathcal{M}(I) \setminus \{0\}$, $-h \neq h$, and this would yield $(-h) \cdot z$ mod $\frac{\hat{M}}{2} = h \cdot z$ mod $\frac{\hat{M}}{2}$ which corresponds to the first case. In total, we obtain $h \cdot z$ mod $\frac{\hat{M}}{2} \neq h' \cdot z$ mod $\frac{\hat{M}}{2}$ for all $h, h' \in \mathcal{M}(I)$, $(|h_1|, \ldots, |h_d|)^\top \neq (|h'_1|, \ldots, |h'_d|)^\top$, implying condition (7).

Remark IV.3. Condition (7) and (10) with $\hat{M} = 2M$ are not equivalent in general. For instance, the generating vector $z := (8, 9)^\top$ and size parameter $M := 72$ from Fig. 2a fulfill condition (7) for $I = I_8^2$ but not condition (10) with $M = 2M$.

We note that there exist special cases where both the conditions (7) and (10) are fulfilled, see e.g. the examples in Fig. 2b and 2c, which fulfill both conditions for $I = I_8^2$.

Remark IV.4. There always exists a reconstructing rank-1 lattice $\Lambda(z, M)$ for $\mathcal{M}(I)$ with even rank-1 lattice size

$$\hat{M} \leq 2 \max \left\{ \frac{2}{3}(|\mathcal{M}(I)|^2 - |\mathcal{M}(I)| + 8), \max_{k \in I} 3||k||_\infty \right\}$$

and consequently a rank-1 Chebyshev lattice $\text{CL}(z, M)$ with size parameter $M := \hat{M}/2$. This result is due to [8, Theorem 2.1] which is a direct consequence of the results from [7].

C. Tent-transformed rank-1 lattices for cosine polynomials

In [12], [13], tent-transformed rank-1 lattices $P_\psi(z, \hat{M}) := \{\phi(jz/\hat{M}) : j = 0, \ldots, \hat{M} - 1\}$, fulfilling a condition equivalent to (10) are used, where $z \in \mathbb{N}_0^d$, $\hat{M} \in \mathbb{N}$ and the tent transform $\phi : [0, 1] \to \mathbb{R}$, $\phi(x) := 1 - |2x - 1|$, is applied component-wise. Then, the exact reconstruction of cosine polynomials $\tilde{p} : [0, 1] \to \mathbb{R}$, $\tilde{p}(x) := \sum_{k \in I} \tilde{a}_k \prod_{i=1}^d \cos(\pi k_i x_i)$, $I \subset \mathbb{N}_0^d$, can be performed by applying a fast Fourier transform to samples at these nodes, cf. [13]. Note that these polynomials $\tilde{p}$ are not algebraic polynomials in general.

V. NUMERICAL RESULTS

Using the algorithm in Fig. 5, we construct rank-1 Chebyshev lattices $\text{CL}(z, M)$ fulfilling condition (7) for the $\ell_1$-ball index sets $I := I_n^d$ for various refinements $n \in \mathbb{N}$ and dimensions $d$. The corresponding size parameters $M$ and oversampling factors $(M + 1)/|I_n^d|$ are shown in Table I. Additionally, we apply [7, Algorithm 1 and 2] to the extended symmetric index sets $\mathcal{M}(I_n^d)$ with the modification that an even rank-1 lattice size $\hat{M} \in 2\mathbb{N}$ is returned. We obtain reconstructing rank-1 lattices $\Lambda(z, M)$ for $\mathcal{M}(I_n^d)$ and consequently rank-1 Chebyshev lattices $\text{CL}(z, M/2)$ fulfilling condition (7) for $I_n^d$ due to Theorem IV.2. For the dimensions $d$
and refinements \( n \) in Table I except the case \( d = 7 \) and \( n = 16 \), these rank-1 Chebyshev lattices are identical to the ones constructed by the algorithm in Fig. 5. In the mentioned case, the algorithm in Fig. 5 yielded a slightly larger size parameter \( M = 21 344 934 \). The reason for this is the greedy search for the generating vector \( z \) with fixed initial size parameter \( M = M_{\text{start}} \) and both approaches returned a distinct generating vector \( z \). If we run the algorithm in Fig. 5 setting \( M_{\text{start}} := 21 254 517 \), then both approaches yield an identical rank-1 Chebyshev lattice.

Moreover, we consider hyperbolic cross index sets \( I := H^d_n \). Again, we apply both algorithms for the construction of rank-1 Chebyshev lattices \( CL(z, M) \) suitable for reconstruction. The results of these construction processes are shown in Table II. We remark that the size parameters \( M \) of the rank-1 Chebyshev lattices \( CL(z, M) \) are distinctly larger for \( d \geq 3 \) when using [7, Algorithm 1 and 2], which itself uses condition (10).

**VI. CONCLUSION**

In this paper, we considered the fast evaluation as well as the fast and exact reconstruction of arbitrary high-dimensional multivariate algebraic polynomials in Chebyshev form. To this end, we used the nodes of rank-1 Chebyshev lattices. Moreover, we presented an algorithm for the construction of rank-1 Chebyshev lattices suitable for reconstruction based on ideas for the CBC construction in the periodic case.

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