Construction of Orthonormal Directional Wavelets based on quincunx dilation subsampling

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Abstract—Using quincunx downsampling for bases and standard downsampling for low-redundancy frames, we construct directional wavelet systems that have the same direction selectivity as shearlets in the first frequency dyadic ring; these are the first step towards the construction of efficient shearlet systems.

I. INTRODUCTION

In image compression and analysis, 2D tensor wavelet schemes are widely used, despite poor orientation selectivity: even for “cartoon images” (piecewise smooth, with jumps occurring along piecewise $C^2$-curves), only horizontal or vertical edges are well represented by tensor wavelets. To address this, several different constructions of directional wavelet systems have been proposed. Shearlet [1], [2] and curvelet [3] systems construct a multi-resolution partition of the frequency domain by applying shear or rotation operators to a generator function in each level; contourlets [4] combine the Laplacian pyramid scheme with a directional filter approach; dual-tree wavelets [5] are linear combinations of 2D tensor wavelets (corresponding to multi-resolution systems) that constitute an approximate Hilbert transform pair.

Among these, shearlets and curvelets have optimal asymptotic rate of approximation for “cartoon images”, due to the parabolic scaling rule in the frequency domain [6], [7]; they have been successfully applied to image denoising [8], restoration [9] and separation [10]. Despite their theoretical potential, the (often high) redundancy of curvelets and shearlets impedes their practical usage. Depending on the generator function and the number of directions, available shearlet and curvelet implementations have redundancy at least 4; moreover, the factor may grow with the decomposition level. Redundancy is useful in image processing tasks such as denoising, restoration and reconstruction, but a non-redundant basis decomposition is preferred in tasks where computation cost is of concern.

Our goal is to construct a directional wavelet system that has similar orientation selectivity to the shearlet system in the first “dyadic frequency ring” but with much less (ideally no) redundancy. This is a first step towards the construction of quasi-shearlet bases, with higher frequency dyadic rings further subdivided according to the parabolic scaling rule.

This work overlaps with that of S. Durand; in particular, the Shannon version of the directional wavelet basis we construct is the same as in [11]. We use a different approach, however, to regularize the frequency behavior (and thus get better decay in the spatial domain): instead of using multi-band filtering with non-uniform sampling, we smooth the Fourier transform of the corresponding wavelet directly, avoiding aliasing.

In what follows, we first set up the framework of a dyadic MRA (3) with quincunx dilation downsampling. Then, we derive two conditions, identity summation and shift cancellation, equivalent to perfect reconstruction in this MRA with critical downsampling. These lead to the classification of regular/singular boundaries of the frequency partition and the corresponding smoothing techniques to improve spatial localization. Next, we introduce a low-redundancy MRA frame (7) that allows better regularity of the directional wavelets. Finally, we compare our and Durand’s directional wavelet constructions and discuss possible extensions of our framework.

II. FRAMEWORK SETUP

We summarize 2D-MRA systems, matrix representations of sublattices of $\mathbb{Z}^2$ and the relation between frequency domain partition and sublattice with critical downsampling.

A. Multi-resolution analysis and critical downsampling

In an MRA, given a scaling function $\phi \in L^2(\mathbb{R}^2)$, s.t. $\|\phi\|^2 = 1$, the base approximation space is defined as $V_0 = \text{span}\{\phi_{0,k}\}_{k \in \mathbb{Z}^2}$, where $\phi_{0,k} = \phi(x-k)$. If $\phi_{0,k} = \delta_{k,k'}$, then $\{\phi_{0,k}\}$ is an orthogonal basis of $V_0$. Moreover, $\phi$ is associated with a scaling matrix $D \in \mathbb{Z}^2 \times \mathbb{Z}^2$ with determinant $|D|$, s.t. the rescaled $\phi_l(x) = |D|^{-1/2}\phi(D^{-1}x)$ is a linear combination of $\phi_{0,k}$. Equivalently, in the frequency domain

$$\hat{\phi}(D^T \xi) = M_0(\xi) \hat{\phi}(\xi),$$

(1)

where $M_0(\xi) = M_0(\xi_1, \xi_2)$, $2\pi$-periodic in $\xi_1, \xi_2$. Hence

$$\hat{\phi}(\xi) = (2\pi)^{-1} \prod_{k=1}^{\infty} M_0(D^{-k} \xi).$$

(2)

The MRA uses the nested approximation spaces $V_l = \text{span}\{\phi(D^{-l}x-k) ; k \in \mathbb{Z}^2\}$, $l \in \mathbb{Z}$. Next, suppose there are wavelet functions $\psi \in L^2(\mathbb{R}^2)$, $1 \leq j \leq J$, and $Q \in \mathbb{Z}^2 \times \mathbb{Z}^2$, s.t. the space $W_l = \bigcup_{j=1}^{J} W_l^j = \bigcup_{j=1}^{J} \text{span}\{\psi_j(D^{-l}x-k) ; k \in Q\mathbb{Z}^2\}$ is the orthogonal complement of $V_l$ with respect to $V_0$. Let $\psi_{l,k}' = |D|^{-l/2}\psi_j(D^{-l}x-k')$; an $L$-level multi-resolution system with base space $V_0$ is then spanned by

$$\{\phi_{L,k}, \psi_{l,k}' ; 1 \leq l \leq L, k \in \mathbb{Z}^2, k' \in Q\mathbb{Z}^2, 1 \leq j \leq J\}.$$  

(3)

As $W_l \subset V_0$, each rescaled wavelet $\psi_j(D^{-l} \cdot)$ is also a linear combination of $\phi_{0,k}$, so that $\exists M_j$ analogous to $M_0$ satisfying

$$\hat{\phi}(D^T \xi) = M_j(\xi) \hat{\phi}(\xi),$$

(4)
In this construction of MRA, the scaling function $\phi$ and all the wavelet functions $\psi^j$ share the same scaling matrix $D$, yet the family of shifted $\phi_k$ is defined on $\mathbb{Z}^2$, whereas the family of shifted $\psi^j_k$ is defined on a sub-integer lattice $Q\mathbb{Z}^2$. Hence the corresponding subsampling matrix of $\phi_{1,k}$ is $D$ and that of $\psi^j_{1,k}$ is $QD$, as in [11]. We haven’t yet imposed any condition on this MRA, or equivalently, on $M$–functions and the subsampling matrices $D$ and $Q$; this comes next.

If the MRA generated by (3) achieves critical downsampling, then $1 + J/|Q|^{-1} = |D|$ ([11]); critical downsampling thus depends only on the subsampling matrices $D$ and $Q$. The space decomposition structure $V_0 = V_1 \bigoplus W_1$ in MRA and (1), (4) require consistency between the $M$–functions and the subsampling matrices $D$ and $Q$.

B. Frequency domain partition and sublattice sampling

Definition. If $\mathcal{L}$ is the lattice generated by $a_1, a_2$, i.e. $\mathcal{L} = \sum_{i=1,2} k_i a_i, k_i \in \mathbb{Z}$, the reciprocal lattice $\mathcal{L}^*$ of $\mathcal{L}$ is the lattice generated by the vectors $b_1, b_2$, s.t. $b_i^T a_j = 2\pi \delta_{ij}$.

Definition. Given a lattice $\mathcal{L}$, a fundamental domain $S$ in $\mathbb{R}^2$ with respect to $\mathcal{L}$, denoted as $S = \mathbb{R}^2 / \mathcal{L}$, is a subset of $\mathbb{R}^2$, such that $\bigcup_{\ell \in \mathcal{L}} (S+\ell) = \mathbb{R}^2$ and $S \cap (S+\ell) = \emptyset$, $\forall \ell \in \mathcal{L}\setminus\{0\}$. A set $S$ is a frequency support of $\mathcal{L}$ if $S = \mathbb{R}^2 / \mathcal{L}^*$.

To build our first example, in which $\hat{\phi}, \hat{\psi}^j$ are indicator functions in $\mathbb{R}^2$, we consider the case where $\mathcal{L} = \mathbb{Z}^2$, $\mathcal{L}^* = 2\pi \mathbb{Z}^2$, and we pick $S_0 = \mathbb{R}^2 / (\mathbb{Z}^2)^*$, the canonical frequency square, $[-\pi, \pi) \times [-\pi, \pi)$. Since $\phi_1, \psi^j_1$ and their shifts span the space $V_0$, $\text{supp}(\hat{\phi}_1)$ and $\text{supp}(\hat{\psi}^j_1)$, together, should thus cover $S_0$. Due to (1) and (4), this is equivalent to $S_0 = \bigcup_{0 \leq j \leq 4} \text{supp}(M_j |_{S_0})$. That is, if $S_1, C_j, 1 \leq j \leq 4$ are the main support of $M_j$, $0 \leq j \leq J$ respectively, then they form a partition of $S_0$. An admissible partition is defined as follows.

Definition. $S_1$ together with $C_j, 1 \leq j \leq J$ is an admissible partition of $S_0$ if and only if $3D, Q \in \mathbb{Z}^{2 \times 2}$, s.t. $S_1 = \mathbb{R}^2 / (D\mathbb{Z}^2)^*$, $C_j = \mathbb{R}^2 / (Q\mathbb{Z}^2)^*$.

To build an orthonormal basis with good directional selectivity, we choose the partition of $S_0$ to be that of the least redundant shearlet system, see Fig.1 left, which is also Example B in [11]. In this partition, $S_0$ is divided into a central square $S_1 = (\frac{3}{2}, \frac{1}{2})^{-1} S_0$ and a ring: the ring is further cut into six pairs of directional trapezoids $C_j$’s by lines passing through the origin with slopes $\pm 1, \pm 3$ and $\pm \frac{3}{2}$. The central square $S_1$ can be further partitioned in the same way to obtain a two-level multi-resolution system, as shown in Fig.1.

This partition is admissible and corresponds to $D = D_2 = (\frac{3}{2}, \frac{3}{2})$ and $Q = (\frac{1}{2}, \frac{1}{2})^{-1}$. The wavelet coefficients are then taken on the dyadic quincunx sublattice $Q D_2 \mathbb{Z}^2$ (see the right panel in Fig.1). In addition, $|D_2| = 4, |Q| = 2$ so that $1 + 6/2 = 4$, and the system has critical downsampling.

III. Parseval frame and bases conditions

In the rest of the paper, we consider only $M$–functions with supports mainly corresponding to the partition of $S_0$ in Fig.1.

As $D$ and $Q$ (dyadic dilation and quincunx subsampling respectively) are determined by the admissibility of this partition, only the $M$–functions need to be designed to construct (3), which is a close analogue of the classical MRA of 1D wavelets. We begin with examining the conditions on $M$–functions such that the system (3) is a Parseval frame.

A. Identity summation and shift cancellation

In MRA, the Parseval frame property is equivalent to the one-layer perfect reconstruction condition: $\forall f \in L_2(\mathbb{R}^2), \sum_k \langle f, \phi_{0,k} \rangle \phi_{0,k} = \sum_k \langle f, \psi^j_{1,k} \rangle \psi^j_{1,k} + \sum_j \sum_k \langle f, \psi^j_{1,k} \rangle \psi^j_{1,k}$. Using (1) and (4) together with the admissibility of the frequency partition, this condition on $\phi$ and $\psi^j$’s yields:

**Theorem 1.** The perfect reconstruction condition holds for (3) iff the following two conditions hold

$$|M_0(\xi)|^2 + \sum_{j=1}^6 |M_j(\xi)|^2 = 1$$

$$\sum_{j=0}^6 M_j(\xi) M_j(\xi + \gamma) = 0, \quad \gamma \in \Gamma \setminus \{0\}$$

$$\sum_{j=1}^6 M_j(\xi) M_j(\xi + \nu) = 0, \quad \nu \in \Lambda \setminus \Gamma$$

where $\Lambda = (Q D_2^*)^*/(\mathbb{Z}^2)^*, \Gamma = (D_2^*)^*/(\mathbb{Z}^2)^*$.

Thm. 1 is a corollary of Prop. 1 and Prop. 2 in [11]. In Thm. 1, Eq. (5) is the identity summation condition, guaranteeing conservation of $l_2$ energy; Eq. (6) is the shift cancellation condition. $\Lambda = \{ (\frac{3}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{3}{2}), (\frac{3}{2}, \frac{3}{2}), (0, 0), (0, \pi), (\pi, 0), (\pi, \pi) \}$, $\Gamma = \{ (0, 0), (0, \pi), (\pi, 0), (\pi, \pi) \}$ are the sets of shifts associated to quincunx-dyadic dilation $QD$ and dyadic dilation $D$ respectively.

B. Extra condition for basis

The system (3) is a Parseval frame by Thm. 1; for it to be an orthonormal basis, $\{\phi_k\}_{k \in \mathbb{Z}^2}$ need to be an orthonormal basis. Because $\phi$ is determined by $M_0$ in (2) (like in 1D MRA), we can find a necessary and sufficient condition on $M_0$ such that (3) is an orthonormal basis. The following theorem is a 2D generalization of Cohen’s 1D theorem in [12].

**Theorem 2.** Assume that $M_0$ is a trigonometric polynomial with $M_0(0) = 1$, and define $\hat{\phi}(\xi)$ as in (2). If $\hat{\phi}(\cdot - k), k \in \mathbb{Z}^2$ are orthonormal, then $\exists K$ containing a neighborhood of 0, s.t. $\forall \xi \in S_0, \xi + 2\pi n \in K$ for some $n \in \mathbb{Z}^2$, and $\inf_{\xi \in K} |M_0(D_2^* \xi)| > 0$. Further, if $\sum_{\gamma \in \Gamma} |M_0(\xi + \gamma)|^2 = 1$, then the inverse is true.

Thm. 2 can be proved similarly to Cohen’s theorem ([12]). Below, we construct $M$–functions imposing only (5) and
of the boundary regularity of
Yet, we shall see that partial direct smoothing is still possible.

To introduce the (slight abuse of) notation generated from these
functions directly. As shown in Proposition 3 in

functions need to be regularized.

have overlapping smoothed boundaries by this \( \nu \).

Left bottom: intersection of \( B(0, \nu) \) and \( B(5, \nu) \) in yellow and \( C(0, \nu) = B(0, \nu) \setminus B(5, \nu) \) in red. Smoothing \( M_0 \) in the red (singular) region is impossible without violating (6). Right: boundary classification, singular (red) and regular (yellow) after similar arguments for all \( \nu \).

(6) and then check if the resulting Parseval frame is an
orthonormal basis by applying Thm. 2 to \( M_0 \).

IV. M-FUNCTION DESIGN AND BOUNDARY REGULARITY

In this section, we define Shannon-type directional orthonor-
mal basis, and then apply direct smoothing to its \( M \)-functions to
improve spatial localization, this leads to a critical analysis of
the boundary regularity of \( S_1 \) and the \( C_j \)'s.

A. Shannon-type wavelets and smoothing

If each \( M_j \) is an indicator function of a piece in the
partition of \( S_0 \), i.e. \( M_0 = \mathbb{1}_{S_1}, M_j = \mathbb{1}_{C_j}, 1 \leq j \leq 6 \),
and we use the boundary assignment of \( C_j \) in the left of
Fig. 1, then the identity summation follows from the frequency
partition, and the shift cancellation holds automatically due to
\( M_j(\xi)\overline{M_j(\xi+\nu)} = 0, \forall j, \nu \neq 0 \). The Shannon-type wavelets generated from these \( M \)-functions form an orthonormal basis.

However, because of the discontinuity of \( M_j \) across the
boundary of its support, the corresponding wavelet has slow
decay in the time domain. In order to improve their spatial
localization, the \( M \) functions need to be regularized.

We take a different regularization approach from Durand's
[11], where three regular quinquex filter banks are con-
structed and then composed to obtain the desired regular quinquex
dyadic filter banks. Here, we smooth the discontinuous bounda-
ries of \( M \)-functions directly. As shown in Proposition 3 in
[11], it is not possible to smooth all the discontinuous bounda-
ries if the \( M_j \) satisfy the perfect reconstruction condition.
Yet, we shall see that partial direct smoothing is still possible.

B. Boundary classification

After smoothing the \( M_j \)'s, the shift cancellation (6) may fail to
hold automatically as \( \text{supp}(M_j) \) and \( \text{supp}(M_j - \nu) \) may
overlap near the smoothed boundaries, see Fig. 2, illustrating
\( M_j(\xi)\overline{M_j(\xi+\nu)} \neq 0 \). For simplicity, we introduce the fol-
lowing notations: let \( B(j, \nu) = \text{supp}(M_j) \cap \text{supp}(M_j - \nu) \) be the support of \( M_j(\xi)\overline{M_j(\xi+\nu)} \) associated to \( M_j \) and shift \( \nu \); let \( C(j, \nu) = B(j, \nu) \setminus \bigcup_{j' \neq j} B(j', \nu) \). In addition, we may introduce the (slight abuse of) notation \( C_0 = S_1 \).

Lemma 1. Shift cancellation (6) can hold for shift \( \gamma \in \Gamma \setminus \{0\} \),
only if \( C(j, \gamma) = \emptyset, \forall 0 \leq j \leq 6 \); Shift cancellation (6) can hold for shift \( \nu \in \Lambda \setminus \Gamma \), only if \( C(j, \nu) = \emptyset, \forall 1 \leq j \leq 6 \).

Proof. Observe that, on \( C(j, \gamma) \), \( M_j[j(\xi)]M_j(j(\xi) + \gamma) \neq 0 \) but
\( M_j[j(\xi)]M_j(j(\xi) + \gamma) \equiv 0 \), \( \forall j' \neq j \), that is (6) doesn’t hold.

Therefore, boundaries that after smoothing make \( C(j, \nu) \)
non-empty are called singular; the rest are regular. We next
provide an explicit boundary classification method:

Proposition 1. \( \forall 1 \leq j \leq 6 \), let \( \text{supp}(M_j) = \overline{C_j} \), then
the boundary of \( C_j \) is \( \partial C_j = \bigcup_{\lambda \in \Lambda \setminus \{0\}} B(j, \lambda) \). The set of
singular boundaries of \( \partial C_j \) is \( \bigcup_{\lambda \in \Lambda \setminus \{0\}} C(j, \lambda) \), whereas
its compliment set is the regular boundary set.

Proof. Since \( \bigcup_{\lambda \in \Lambda}(C_j + \lambda) = S_0 \), \( \partial C_j \subset \bigcup_{\lambda \in \Lambda \setminus \{0\}} (C_j + \lambda) \). Therefore, \( \partial C_j \subset \bigcup_{\lambda \in \Lambda \setminus \{0\}} B(j, \lambda) \). On the other hand, \( B(j, \lambda) \subset \partial C_j \), \( \forall \lambda \in \Lambda \setminus \{0\} \), hence the union of them is a subset of \( \partial C_j \). It follows that \( B(j, \lambda) \) form a partition of the boundary \( \partial C_j \). The partition of \( \partial C_j \) into singular and regular boundaries follows from Lemma 1.

The case of \( \partial S_1 \) is similar where \( B(0, \gamma), \gamma \in \Gamma \setminus \{0\} \) are considered. We use the notation \( B_s(j, \nu) \), \( C_s(j, \nu) \) for the special case \( \text{supp}(M_j) = \overline{C_j} \) hereafter. The boundary classification based on Proposition 1 is shown in the right of Fig. 2, where the boundaries on the four corners of both \( S_0 \) and \( S_1 \) are singular: smoothing is then not allowed there.

Proposition 2. Let \( C = C_s(j_1, \nu_1) \cap C_s(j_2, \nu_2) \), if \( M_{j_1}, M_{j_2} \) satisfy (5), (6), then \( |M_{j_1}| = \mathbb{1}_{C_{j_1}}, |M_{j_2}| = \mathbb{1}_{C_{j_2}} \) on \( C \).

Proof. Suppose the common singular boundary \( C \) is non-
empty and observe that \( C \subset (C_{j_1}) \cap (C_{j_2}) \). Since \( M_{j_1} \) cannot be smoothed on \( C_s(j_1, \nu_1) \), \( |M_{j_1}| = 0 \) on \( C_s(j_1, \nu_1) \setminus C_{j_1} \), and
(5) implies that \( |M_{j_2}| = 1 \) there, or equivalently \( |M_{j_2}| = \mathbb{1}_{C_{j_2}} \) on \( C \). Similarly, \( |M_{j_1}| = \mathbb{1}_{C_{j_1}} \) on \( C \).

Prop 2 shows that if \( M_j \) and \( M_j \) have common singular boundaries, then both will have a discontinuity across those boundaries. For example, \( C_s(0, (\pi, 0)) \cap C_s(4, (\pi/2, \pi/2)) = (\pi/2, (\pi/2, \pi/2)) \), hence \( M_1 \) and \( M_4 \) both are discontinuous at \( (\pi/2, (\pi/2, \pi/2)) \). All the singular boundaries related to (3) are such “double” singular boundaries.

C. Pairwise smoothing of regular boundary

The regular boundaries of both \( C_{j_1} \) and \( C_{j_2} \), with adjacent supports consist of \( B_s(j_1, \nu) \cap B_s(j_2, \nu) \), which we denote by
the triple \( (j_1, j_2, -\nu) \). The following proposition shows that the regular boundaries \( (j_1, j_2, \nu) \) can be paired according to shift pairs \( (\nu, -\nu) \), and the boundaries must be smoothed pairwise within their \( \epsilon \)-neighborhood, \( B_s(j_1, j_2, \nu) \) at \( j_1, j_2, -\nu \).

Proposition 3. Given \((j_1, j_2, \nu) \neq \emptyset \), then \((j_1, j_2, -\nu) \neq \emptyset \). In addition, let \( B = B_s(j_1, j_2, \nu) \cup B_s(j_1, j_2, -\nu) \). Then the identity summation and shift cancellation conditions hold if

(i) \( M_j = \mathbb{1}_{C_{j_1}} \), on \( \emptyset, j \neq j_1, j_2 \)
(ii) \( M_j = \mathbb{1}_{C_{j_1}}, M_j = \mathbb{1}_{C_{j_2}} \), on \( \mathbb{C}^c \)
and on \( \mathbb{B} \) the following hold
(iii) \( |M_{j_1}|^2 + |M_{j_2}|^2 = 1, \)
(iv) \( \sum_{j_1,j_2} M_{j_1}(\cdot)M_{j_2}(\cdot + \nu) = 0 \), \( \nu = \pm \nu \).
Proof. We first show that \((j_1, j_2, -\nu) \neq \emptyset\). By definition, \(B_s(j_1, \nu) = \text{supp}(M_{j_1}) \cap (\text{supp}(M_{j_1}) - \nu) = ((\text{supp}(M_{j_1}) + \nu) \cap \text{supp}(M_{j_1})) - \nu = B_s(j_1, -\nu) - \nu\). Rewrite \((j_1, j_2, \nu)\) by \(B_s(j_1, -\nu)\) and \(B_s(j_2, -\nu)\), we have \((j_1, j_2, -\nu) = (j_1, j_2, \nu) + \nu\), hence it’s non-empty.

Because \((j_1, j_2, \pm \nu) \subset (\partial C_{j_1} \cap \partial C_{j_2})\) and \((\bigcup_{j \neq j_1, j_2} C_j) = S_0\), \((\bigcup_{j \neq j_1, j_2} C_j) \cap \mathcal{B} = \emptyset\). Therefore, smoothing of \(M_{j_1}\) and \(M_{j_2}\) in \(\mathcal{B}\) doesn’t impact the region where other \(M_j\)'s are supported.

We then show that the cancellation conditions (6) hold for all shifts. Condition (i) and (ii) imply that \(B(j, \nu) = \emptyset\), \(\forall j, \nu \neq \pm \nu\), hence (6) hold for \(\nu \neq \pm \nu\), (i) implies \(B(j, \nu) = \emptyset\), \(\forall j \neq j_1, j_2\), so then (6) is equivalent to (iv). The identity summation (5) holds due to (i), (ii) and (iii). □

By Prop. 3 we can smooth some pairs of regular boundaries starting from the Shannon-type directional wavelets with the simplified conditions (iii), (iv) and (v); (i) and (ii) can be removed as long as the initial \(M_j\) satisfy (5) and (6) and every \(\xi \in S_0\) is not covered by more than two \(M\) functions. We can thus smooth regular boundaries pairwise, one by one.

The next proposition gives an explicit design of \((M_{j_1}, M_{j_2})\) satisfying the simplified conditions (iii)-(iv) in Proposition 3.

Proposition 4. Let \(\Omega \subset S_0\), given \(M_{j_1}, M_{j_2} \neq 0\) continuous on \(\Omega \cup (\Omega + \nu)\), satisfying the following conditions

(i) \(\sum_{j_1, j_2} M_{j_1}(\xi) M_{j_2}(\xi + \nu) = 0\) on \(\Omega\)

(ii) \(\sum_{j_1, j_2} M_{j_1}(\xi) \overline{M_{j_2}(\xi + \nu)} = 0\) on \(\Omega \cup (\Omega + \nu)\)

(iii) \(M_{j_1}(\xi) M_{j_2}(\xi) = 0\) on \(\partial \Omega\);

then \(|M_{j_1}(\xi)| = |M_{j_2}(\xi + \nu)|, \quad |M_{j_2}(\xi)| = |M_{j_1}(\xi + \nu)|\).

Furthermore, if \(M_j = e^{i\xi \cdot \eta_1} M_{j_2}, j = j_1, j_2, \Omega, \) where \(M_j\) is a real-valued function, \(e^{i\xi \cdot (\eta_1 - \eta_2)} = -1\), and \(M_{j_1}(\xi) = M_{j_2}(\xi - \nu), \quad M_{j_2}(\xi) = M_{j_1}(\xi - \nu)\), on \(\Omega + \nu\), then (i) holds.

Proof. To prove the necessary condition, note that (i) implies \(|M_{j_1}(\xi)|^2 M_{j_2}(\xi + \nu)|^2 = |M_{j_2}(\xi)|^2 M_{j_1}(\xi + \nu)|^2\); the condition then follows from (ii). For the sufficient construction, check by directly substituting the construction into (i).

Proposition 4 breaks down the design of \((M_{j_1}, M_{j_2})\) into a pair of real functions \((M_{j_1}, M_{j_2})\) on \(B_s(j_1, j_2, \nu)\) and two vectors \(\eta_1, \eta_2\); then \((M_{j_1}, M_{j_2})\) on \(B_s(j_1, j_2, -\nu)\) are automatically determined. The only constraint on \((M_{j_1}, M_{j_2})\) for (ii) in Proposition 4 to hold is that on \(B_s(j_1, j_2, \nu), \sum_{j_1, j_2} |M_j(\xi)|^2 = 1\), which is easy to be satisfied. We may construct all local pairs of \((M_{j_1}, M_{j_2})\) separately, and put together afterwards different pieces of each \(M_j\) located in different regular boundary neighborhoods \(B_s(j, j', \nu)\).

The next proposition gives one solution, easy to verify.

Proposition 5. Applying Proposition 4 to all regular boundaries requires a set of phases \(\{\eta_1, \eta_2\}_{0=0}^6\), s.t.

\(e^{i\xi \cdot (\eta_1 - \eta_2)} = -1, \quad \forall (j_1, j_2, \nu) \in \Delta,\)

where \(\Delta = \{(0, 2, (0, \pi)), (0, 5, (\pi, 0)), (1, 3, (\pi, 0)), (4, 6, (0, \pi)), (1, 6, (\pi/2, 3\pi/2)), (2, 3, (\pi/2, 3\pi/2)), (4, 5, (\pi/2, 3\pi/2)), (3, 4, (\pi/2, \pi/2)), (1, 2, (\pi/2, \pi/2)), (5, 6, (\pi/2, \pi/2))\}\).

The following is a (non-unique) solution:

\(\eta_0 = (0, 0), \quad \eta_1 = (0, 0), \quad \eta_2 = (1, 1), \quad \eta_3 = (1, -1), \quad \eta_4 = (0, 2), \quad \eta_5 = (1, 1), \quad \eta_6 = (-1, 1).\)

To summarize, Proposition 4 and 5 introduce the following regular boundary smoothing scheme for the \(M\) functions:

1. First, set \(M_j = 1_{C_j}\); then smooth these across a pair of regular boundaries \((j_1, j_2, \pm \nu)\) following steps 2, 3.
2. On \(B_s(j_1, j_2, \nu)\),
   design \((M_{j_1}, M_{j_2})\), s.t. \(\sum_{j_1, j_2} |M_{j_1}|^2 = 1\).
3. On \(B_s(j_1, j_2, -\nu)\),
   let \(M_{j_1}(\xi) = M_{j_2}(\xi - \nu), \quad M_{j_2}(\xi) = M_{j_1}(\xi - \nu)\)
4. Repeat step 2 and 3 for all \((j_1, j_2, \nu) \in \Delta\).
5. \(M_j(\xi) = e^{i\xi \cdot \eta_1} M_j(\xi),\) on \(S_0\), with the \(\eta_j\) of Prop. 5.

We apply this to smooth all the regular boundaries except those on the boundary of \(S_0\). Near a regular boundary \(B_s(j, j', \nu)\), the change of \(|M_j|\) from 0 to 1 depends on \(M_j\); the contour of stop-band/pass-band is the boundary of level set \(|M_j(\xi) = 0|/\{M_j(\xi) = 1\}\). Fig. 3 shows our design of the stop-band/pass-band contours of regular boundaries (5, 6, \((\pi/2, \pi/2)) and (0, 5, (\pi, 0)). The contours intersect only at the vertices of \(C_5\), e.g. \(\text{supp}(M_5) \cap \text{supp}(M_6) \cap \text{supp}(M_0)\) contains just one point. Moreover, we set \(M_4\) to be symmetric with respect to the origin near both regular boundaries.

The contours related to other regular boundaries are designed likewise to achieve the best symmetry; the corresponding wavelets are real. Fig.3 (right) shows the frequency support of directional wavelets generated by such design; Fig.4(a) shows the wavelets and scaling function in space domain. One easily checks (using Thm. 2) that this is an orthonormal basis.

Although the wavelets orient in six directions, they are not very well localized spatially, due to the singular boundaries on the corners of the low-frequency square \(S_1\), where the discontinuity in the frequency domain is inevitable. The lack of smoothness at the vertices of \(M_2\) and \(M_3\) could possibly be avoided by using a more delicate (but more complicated) design around the vertices \((\pm \pi/2, \pm \pi/6)\) allowing triple overlapping of \(M\)-functions.

Allowing a bit of redundancy (abandoning critical down-sampling), we show next how to construct a frame with low redundancy that has much better spatial localization.

V. LOW-REDUNDANCY FRAME CONSTRUCTION

Consider the \(L\)-level directional wavelet MRA system

\[\{\phi_{L-k}, \psi_{l,k}, l \leq L, k, k' \in \mathbb{Z}, 1 \leq j \leq J\}.\]
where \( \phi, \psi \) satisfy (1) and (4) as before. Instead of taking the quincunx dyadic subsampling of directional wavelet coefficients of (3), a dyadic subsampling is taken instead. A 1-level MRA frame (7) has redundancy \( \frac{1}{16} + \frac{1}{4} = 1/4 + 6/4 = 7/4 \), and the redundancy for any \( L \)-level MRA frame doesn’t exceed \( \frac{1}{4^L - 1} \). We have now

**Theorem 3.** Set \( \Gamma = (D\mathbb{Z}^2)^*/(\mathbb{Z}^2)^* \). The perfect reconstruction condition holds for (7) iff the following both hold

\[
|M_0(\xi)|^2 + \sum_{j=1}^{6} |M_j(\xi)|^2 = 1 \\
\sum_{j=0}^{6} M_j(\xi)\tilde{M}_j(\xi + \gamma) = 0, \quad \gamma \in \Gamma \setminus \{0\}
\]

Thm. 3 can be proved analogously to Thm. 1, with fewer shift cancellation constraints now. We can define singular boundaries as before, but only \( \{B(j, \gamma)\}_{\gamma \in \Gamma \setminus \{0\}} \) need to be considered, which results in fewer singular boundaries \( \{C_s(j, \gamma)\}_{\gamma \in \Gamma \setminus \{0\}} \) and no “double” singular boundaries now.

This means that even though \( \text{supp}(M_0) \) still cannot be extended outside of the four corners of \( S_1 \) due to \( C_s(0, (0, 0)) \) and \( C_s(0, (0, \pi)) \), \( M_1 \) can penetrate into the inside of \( S_1 \) because \( C_s(1, (\pi, 2, 3\pi/2)) \) is not a singular boundary in (7). The same is true for \( M_3, M_4 \) and \( M_6 \). This makes smoothing the boundaries of \( M_0 \) inwards possible without violating (5), see Fig. 4(c). At the price of double redundancy, we obtain directional wavelets with much better spatial localization; see Fig. 4(d)(e): the discontinuities of a directional wavelets basis in the frequency domain around the singular boundaries can be removed in a low redundant direction wavelet tight frame.

**VI. DISCUSSION AND FUTURE WORK**

We propose two directional wavelet MRA systems (3) and (7) such that the directional wavelets constructed in the section IV and V characterize 2D signals in six equi-angled directions. The orthonormal basis we construct has better frequency localization than the one constructed by Durand in [11] (see Fig. 3 and 4(b)(c)), but has long tails in certain spatial directions, unavoidable because of “double” singular boundaries. By doubling the redundancy as in (7), one overcomes this drawback, and designs \( M - \)functions in (7) that are \( C^\infty \) and that lead to spatially well localized directional wavelets. Furthermore, these wavelets are well localized in the frequency domain such that \( \text{supp}(M_j) \) is convex and \( 2\epsilon \) s.t.

\[
\sup_{\xi \in \text{supp}(M_j)} \inf_{\xi' \in C} ||\xi' - \xi|| < \epsilon, \quad 0 \leq j \leq 6.
\]

This desirable condition is hard to obtain by multi-directional filter bank assembly of several elementary filter banks.

Both directional wavelet systems (3) and (7) have the same directional selectivity across all the scales. To obtain optimal sparse approximation of cartoon-like images, this is not sufficient for finer scales; instead, a parabolic scaling law of subdivision in frequency domain is needed, which both shearlet and curvelet systems obey[6], [7].

The Shannon-type construction of (3) can be easily extended to have parabolic scaling by dividing each pair of directional trapezoid \( C_0 \) in half by a radial line going through the origin; the corresponding subsampling matrix then becomes \( QDP \), where \( P = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) for horizontal trapezoid pairs, and \( P = \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right) \) for vertical trapezoid pairs; see also [11].

We plan to analyze boundary regularity and apply the smoothing technique using Prop. 4 based on Thm. 1 and Thm. 3, to subdivide our regularized directional wavelet systems and thus construct quasi-shearlet systems.

**REFERENCES**


