On the Strong Divergence of Hilbert Transform Approximations from Sampled Data

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Abstract—It is known that every linear method which determines the Hilbert transform from the samples of the function diverges (weakly). This paper presents strong evidence that all such methods even diverge strongly. It is shown that the common approximation method derived from the conjugate Fejér means diverges strongly, and that all reasonable approximation methods with a finite search horizon diverge strongly. Moreover, the paper discusses the relation between strong divergence and the existence of adaptive approximation methods.

Index Terms—Adaptive signal processing, Hilbert transformation, Sampled data, Strong divergence

I. INTRODUCTION

Let $T : B_1 \to B_2$ be a bounded linear operator between two Banach spaces $B_1$ and $B_2$. An important problem in analysis with many applications in engineering and science is to approximate $T$ by a sequence $\{T_N\}_{N \in \mathbb{N}}$ of finite-rank, linear, bounded operators $B_1 \to B_2$. The question is then whether the sequence $\{T_N f\}_{N \in \mathbb{N}}$ converges to $T f$ for every $f \in B_1$. There are many important examples where $T f$ converges to $T f$ for all $f$ in a dense subset $B_0$ of $B_1$ but it fails to converge for all $f \in B_1$. In many cases such negative results are stated as:

$$\limsup_{N \to \infty} \|T_N f - T f\|_{B_2} = \infty \quad \text{for some } f \in B_1. \tag{1}$$

Results as in (1) are usually proven by showing that the norms $\|T_N\|_{B_2 \to B_2}$ are not uniformly bounded. Then (1) follows immediately from the theorem of Banach-Steinhaus [1]. Moreover, the Banach-Steinhaus Theorem states additionally that the set of all $f \in B_1$ for which (1) holds is a residual set [2].

Example (Trigonometric series): Let $B_1 = B_2 = C(T)$ be the set of all continuous functions on $T = [0, 2\pi]$ equipped with the maximum norm, and let $T = I_C$ be the identity operator on $C(T)$. Set

$$(D_N f) = \sum_{k=0}^{N-1} f\left(\frac{k \pi}{N}\right) D_N\left(t - \frac{k \pi}{N}\right), \quad N \in \mathbb{N}$$

with the Dirichlet kernel

$$D_N(\tau) = \frac{\sin([N + 1/2] \tau)}{\sin(\tau/2)}.$$ 

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Then it is well known that (1) holds for some $f^* \in C(T)$. More precisely, there is a residual set $D \subset C(T)$ such that $\sup_{N \in \mathbb{N}} \|D_N f^* - f^*\|_{\infty} = \infty$ for every $f^* \in D$.

However, a divergence result as in (1) does not exclude the possibilities that

$$\inf_{N \in \mathbb{N}} \|T_N f - T f\|_{B_2} < \infty \quad \text{or} \quad \liminf_{N \to \infty} \|T_N f - T f\|_{B_2} = 0 \tag{2}$$

for all $f \in B_1$, or for all $f$ from a residual set in $B_1$. So there may exist subsequences $\{N_k\}_{k \in \mathbb{N}}$ such that $T_{N_k} f$ converges to the desired function $T f$ as $k \to \infty$. If such a subsequence exists, it will generally depend on the actual function $f$ itself [3]. So the question is whether it is always possible to adapt a given approximation method $\{T_N\}_{N \in \mathbb{N}}$, with property (1), to the actual function $f$ by choosing an appropriate subsequence $\{N_k(f)\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \to \infty} \|T_{N_k(f)} f - T f\|_{B_2} = 0. \tag{3}$$

To investigate such questions, the notion of strong divergence was introduced in [4].

Definition (Strong divergence): Let $B_1$ and $B_2$ be Banach spaces, and let $\{T_N\}_{N \in \mathbb{N}}$ be a sequence of bounded linear operators $T_N : B_1 \to B_2$. We say that $\{T_N\}_{N \in \mathbb{N}}$ diverges strongly if

$$\lim_{N \to \infty} \|T_N f\|_{B_2} = \infty \quad \text{for some } f \in B_1. \tag{4}$$

Clearly, if $\{T_N\}_{N \in \mathbb{N}}$ diverges strongly then none of the situations in (2) can occur for all $f \in B_1$, and so it will not be possible to find for every $f \in B_1$ a subsequence $\{N_k(f)\}_{k \in \mathbb{N}}$ such that (3) holds.

Let $\{T_N\}_{N \in \mathbb{N}}$ be an approximation method which satisfies (1). If $\{T_N\}_{N \in \mathbb{N}}$ diverges strongly, then no adaptive method exists which always converges to the desired result. However, it is shown in [5] that if $\{T_N\}_{N \in \mathbb{N}}$ is a sequence with satisfies (1) and which diverges strongly, then this strong divergence can only occur on a meager set, i.e.

$$\{ f \in B_1 : \limsup_{N \to \infty} \|T_N f - T f\|_{B_2} = \infty \quad \text{and} \quad \liminf_{N \to \infty} \|T_N f - T f\|_{B_2} = 0 \}$$

is a residual set in $B_1$. Thus, for any approximation sequence $\{T_N\}_{N \in \mathbb{N}}$, it is possible to find for almost every
$f \in \mathcal{B}$ a subsequence $\{T_N(f)\}_{k \in \mathbb{N}}$ such that $T_N(f)$ converges to $Tf$. So one can achieve convergence on a residual set by adapting $\{T_N\}_{N \in \mathbb{N}}$ to the actual function.

Because of this close relation between the strong divergence of approximation series and the existence of adaptive approximation methods, it is of some importance to investigate the strong divergence for concrete and practically relevant approximation methods. One of the first investigations in this direction dates back to 1941, where Paul Erdős investigated Lagrange interpolation on Chebyshev notes [6]. He tried to show that the Lagrange interpolation on $C(T)$ diverges strongly. However, he observed later [7] that his proof was erroneous. Since he could not correct the proof, his question whether the Lagrange interpolation diverges strongly seems still to be open. Recently, [4] showed that the Shannon sampling series $S_N$ as a mapping from the Paley-Wiener space $\mathcal{P}W^1$ of bandlimited functions into Bernstein space $\mathcal{B}_\infty$ of bounded bandlimited functions diverges strongly, and [5] investigated strong divergence of system approximations for bandlimited functions.

II. Problem Statement and General Notations

This contribution investigates the strong divergence for a large class of approximation methods of the Hilbert transform. Let $f \in L^1(T)$ be a Lebesgue integrable function on the interval $T := [0, 2\pi]$. Its conjugate function $\tilde{f}$ is given by the Hilbert transform of $f$:

$$\tilde{f}(t) = (Hf)(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\epsilon \leq \tau \leq 2\pi - \epsilon} \frac{f(\tau + t) - f(\tau - t)}{\tan(\tau/2)} \, d\tau$$

where the integral on the right exists for almost all $t \in T$, see e.g., [8, Sect. III.1]. This transformation plays an important role in many different areas such as signal processing, communications, control theory and physics [9, 10].

The Banach space of all continuous functions $f$ on $T$ with $f(2\pi) = f(0)$ equipped with the norm $\|f\|_\infty = \max_{t \in T} |f(t)|$ is denoted by $C(T)$, and $C^\infty(T)$ is the dense subset of all infinitely differentiable $f \in C(T)$. We consider the Hilbert transform on the Banach space:

$$\mathcal{B} := \{ f \in C(T) : \tilde{f} = Hf \in C(T) \}$$

of all $f \in C(T)$ with an continuous conjugate $\tilde{f} = Hf$, equipped with the norm $\|f\|_B := \max_{t \in T} |f(t)|$. We want to uniformly approximate the conjugate function $\tilde{f}$ of any $f \in \mathcal{B}$. In general, this is possible, because it is known [11] that the Fejér series

$$(F_Nf)(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) \hat{f}_n e^{int},$$

wherein $\hat{f}_n$ is the $n$th Fourier coefficient of $f$, converges uniformly to $f$ for every $f \in C(T)$. Consequently for every $f \in \mathcal{B}$, we have by the linearity of $H$

$$\|HF_Nf - \tilde{f}\|_\infty \leq \|F_Nf - \tilde{f}\|_\infty = \|F_Nf - \tilde{f}\|_\infty \to 0$$
as $N \to \infty$. However, to implement this approximation method, one needs to determine the exact Fourier coefficients of $f$. This is often not possible, because of the numerical integration involved in determining $\hat{f}_n$. Instead, we focus on approximation methods which are based on samples of $f$, as described in the next section.

III. Approximation Sequences

Our goal is to approximate the Hilbert transform (4) of an $f \in \mathcal{B}$ by a sequence $\{H_Nf\}_{N \in \mathbb{N}}$ where $H_N : \mathcal{B} \to \mathcal{B}$ are finite-rank, bounded, linear operators, which are computationally feasible, i.e. which are based on a finite number of samples of $f$. More precisely, we require that $\{H_N\}_{N \in \mathbb{N}}$ has the following three natural properties:

(A) Concentration on a finite sampling set: For each $N \in \mathbb{N}$ there is a finite set $A_N = \{\lambda_{n,N}: n = 1, \ldots , M_N\}$ with $\lambda_n \in T$ such that $f(\lambda) = g(\lambda)$ for all $\lambda \in A_N$ implies that $(H_Nf)(t) = (H_Ng)(t)$ for all $t \in T$.

(B) Convergence on a dense subset: We have

$$\lim_{N \to \infty} \|H_Nf - \tilde{f}\|_\infty = 0 \quad \text{for all} \quad f \in C^\infty(T).$$

(C) Generated by a sampling series: To the sequence $\{H_N\}_{N \in \mathbb{N}}$ there corresponds a sequence of approximation operators $A_N : \mathcal{B} \to \mathcal{B}$, $N \in \mathbb{N}$ such that

$$\lim_{N \to \infty} \|A_Nf - f\|_\infty = 0 \quad \text{for all} \quad f \in \mathcal{B}$$

and such that $H_Nf = A_Nf$ for all $N \in \mathbb{N}$.

Remark: A sequence $\{H_N\}_{N \in \mathbb{N}}$ of linear operators has property (A) if and only if to every $N \in \mathbb{N}$ there exists a finite set $A_N = \{\lambda_{1,N}, \lambda_{2,N}, \ldots , \lambda_{M_N,N}\}$ with $M_N \in \mathbb{N}$ and $\lambda_{n,N} \in T$ and a set of functions $\{h_{n,N}: n = 1, \ldots , M_N\}$ in $B$ such that

$$(H_Nf)(t) = \sum_{n=1}^{M_N} f(\lambda_{n,N}) h_{n,N}(t) \quad \text{for all} \quad f \in \mathcal{B}.$$\

Moreover, by the linearity of $H_N$, it is clear that the operators $A_N$ of property (C) have the form

$$(A_Nf) = \sum_{n=1}^{M_N} f(\lambda_{n,N}) a_{n,N}(t), \quad t \in T$$

with $a_{n,N} \in \mathcal{B}$ and such that $h_{n,N} = H a_{n,N}$.

The next lemma collects properties of our approximation sequences $\{H_N\}$ which frequently used subsequently.

**Lemma 1:** Let $\{H_N\}_{N \in \mathbb{N}}$ be a sequence with properties (A) and (B), then the following statements are equivalent.

(i) The sequence $\{H_N\}_{N \in \mathbb{N}}$ has property (C).

(ii) For all $f \in C(T)$, we have $\lim_{N \to \infty} \|A_Nf - f\|_\infty = 0$.

(iii) There exists a constant $C_1 > 0$ such that

$$\sup_{f \in \mathcal{B}, \|f\|_B \leq 1} \|A_Nf\|_\infty \leq C_1 \quad \text{for all} \quad N \in \mathbb{N}.$$\

(iv) There exists a constant $C_2 > 0$ such that

$$\sup_{f \in C(T), \|f\|_\infty \leq 1} \|A_Nf\|_\infty \leq C_2 \quad \text{for all} \quad N \in \mathbb{N}.$$\

The proofs of our results are based on the following lemma. It shows that to every $f \in C(T)$ there exists a $\psi \in \mathcal{B}$ which is infinitely differentiable and such that $f$ and $\psi$ coincide on the finite sampling set $\Lambda$ and such that the norm of $\psi$ is at most a multiplicative constant of $\|f\|_\infty$. 

...
and where the constant does not depend on \( f \). A result similar to Lemma 2 can be found in [12]. The proof in [12] can be adapted to prove Lemma 2. Therefore, the proof of Lemma 2 is omitted here.

**Lemma 2:** There exists a universal constant \( C_3 > 0 \) such that for every finite set \( \Lambda = \{\lambda_1, \ldots, \lambda_N\} \) with \( \lambda_n \in \mathbb{T} \) the following statement is true: To every \( f \in C(\mathbb{T}) \) there exists a function \( \psi \in \mathbb{B} \) such that

\[
\begin{align*}
(i) & \quad \psi(\lambda) = f(\lambda) \quad \text{for all} \quad \lambda \in \Lambda. \\
(ii) & \quad \|\psi\|_{\infty} \leq \|f\|_{\infty} \quad \text{and} \quad \|\psi\|_{\infty} \leq C_3 \|f\|_{\infty}. \\
(iii) & \quad \psi \in C^\infty(\mathbb{T}).
\end{align*}
\]

**Remark:** Note that \( C_3 \) does not depend on \( \Lambda \) or \( f \). Moreover, (ii) implies \( \|\psi\|_{\mathbb{B}} \leq C_3 \|f\|_{\infty} \) with \( C_3 = \max(1, C_3) \).

It was shown in [12] that every sequence \( \{H_N\}_{N \in \mathbb{N}} \) with properties (A), (B), and (C) diverges weakly on \( \mathbb{B} \). More precisely, the following result was proven.

**Theorem 3:** Let \( \{H_N\}_{N \in \mathbb{N}} \) be an operator sequence with properties (A), (B), and (C). The set of all \( f \in \mathbb{B} \) for which

\[
\limsup_{N \to \infty} \|H_N f\|_{\infty} = \infty
\]

is a residual set in \( \mathbb{B} \).

We want to investigate whether such sequences diverge even strongly on \( \mathbb{B} \), i.e., whether it is possible to adapt these sequences to the actual function \( f \) to achieve convergence.

**IV. EXAMPLES OF STRONG DIVERGENCE**

We start our investigations with the observation that any approximation sequence \( \{H_N\}_{N \in \mathbb{N}} \) with properties (A), (B), and (C) diverges strongly on \( C(\mathbb{T}) \). Because of space constrains, the proof is omitted.

**Theorem 4:** There exists a residual set \( \mathcal{D} \subset C(\mathbb{T}) \) such that for every sequence \( \{H_N\}_{N \in \mathbb{N}} \) with properties (A), (B), and (C), we have

\[
\lim_{N \to \infty} \|H_N f\|_{\infty} = \infty \quad \text{for all} \quad f \in \mathcal{D}.
\]

**Remark:** Note that \( \mathcal{D} \) does not depend on the particular sequence \( \{H_N\}_{N \in \mathbb{N}} \) of operators but it is universal in the sense that \( \mathcal{D} \) is the same for all possible sequences \( \{H_N\} \).

Since \( \mathbb{B} \subset C(\mathbb{T}) \), Theorem 4 does not necessarily apply to \( \mathbb{B} \). Although we believe that every approximation procedure with properties (A), (B) and (C) diverges strongly on \( \mathbb{B} \), we are not yet able to prove such a general result. However, we will show that one particular series diverges strongly on \( \mathbb{B} \). To this end, we consider the sampling series

\[
(F_N f) = \sum_{k=0}^{N-1} f(k - k_0) F_N \left( t - k \frac{2\pi}{N} \right), \quad N \in \mathbb{N}
\]

with the Fejér kernel

\[
F_N(\tau) = \left( \frac{\sin(N\tau/2)}{N \sin(\tau/2)} \right)^2.
\]

It follows from the Theorem of Fejér [13] that \( F_N f \) converges uniformly to \( f \) for all \( f \in C(\mathbb{T}) \), i.e.,

\[
\lim_{N \to \infty} \|F_N f - f\|_{\infty} = 0 \quad \text{for all} \quad f \in C(\mathbb{T}).
\]

Based on \( \{F_N\}_{N \in \mathbb{N}} \), we now define the operator sequence \( \{H_N^F\}_{N \in \mathbb{N}} \) by \( H_N^F := HF_N \). This yields

\[
(H_N^F f)(t) = \sum_{k=0}^{N-1} f(k - k_0) \tilde{F}_N \left( t - k \frac{2\pi}{N} \right), \quad t \in \mathbb{T}.
\]

with the conjugate Fejér kernel

\[
\tilde{F}_N(\tau) = \frac{1}{N} \left( \frac{1}{\tan(\tau/2)} - \frac{\sin(N\tau)}{2N \sin^2(\tau/2)} \right).
\]

By this construction, it is easy to verify that \( \{H_N^F\}_{N \in \mathbb{N}} \) has property (A), (B), and (C). The divergence behavior of \( \{H_N^F\}_{N \in \mathbb{N}} \) is governed by the properties of the kernel (6) as given by the next lemma (see [11] for a proof).

**Lemma 5:** Let \( \tilde{F}_N \) be the \( N \)-th order conjugate Fejér kernel given by (6). Then for all \( n \in \mathbb{N} \)

\[
(i) \quad \tilde{F}_N(\tau) \leq 0 \quad \text{for all} \quad 0 < \tau < \pi \]

\[
(ii) \quad C(N) := \sum_{k=0}^{N-1} \tilde{F}_N \left( \pi - k \frac{2\pi}{N} \right) \geq \frac{2}{\pi} \log(N + 1) - C_{\tilde{F}} \]

with a positive constant \( C_{\tilde{F}} \) independent of \( N \).

Now, we go to show that the particular approximation method (5) diverges strongly on \( \mathbb{B} \). Theorem 6.

**Theorem 6:** Let \( \{H_N^F\}_{N \in \mathbb{N}} \) be the sequence defined in (5). There exists a function \( f_0 \in \mathbb{B} \) such that

\[
\lim_{N \to \infty} (H_N^F f_0)(\pi) = \infty.
\]

**Remark:** Theorem 6 not only shows the strong divergence of \( \{H_N^F\} \) on \( \mathbb{B} \), i.e., \( \lim_{N \to \infty} \|H_N^F f\|_{\mathbb{B}} = \infty \). It even shows that \( \{H_N^F f\} \) diverges strongly at the fixed point \( \pi \in \mathbb{T} \).

**Proof:** Let \( f_0 \) be the function defined by

\[
 f_0(t) = \begin{cases} 
 1 & : \quad t \in (0, \pi) \\
 0 & : \quad t \notin (0, \pi) 
\end{cases},
\]

and let \( L_N \in \mathbb{N} \) be the largest number so that \( L_N \frac{2\pi}{N} < \pi \).

For any \( M \in \mathbb{N} \) let

\[
T_M = \bigcup_{N=3}^{M} \Lambda_N \quad \text{with} \quad \Lambda_N = \{ k \frac{2\pi}{N}, \quad 1 \leq k \leq L_N \}
\]

be the set of all sampling points in the interval \((0, \pi)\) which are used by at least one of the approximation operators \( H_N^F \) with degree \( N = 3, 4, \ldots, M \).

For any \( k \in \mathbb{N} \), there is a function \( \phi_k \) with the following properties (cf. Lemma 2): \( \|\phi_k\|_{\infty} = 1, \|\phi_k\|_{\infty} \leq C_3 \) and

\[
\phi_k(\lambda) = f_0(\lambda) \quad : \quad \text{for all} \quad \lambda \in T_k \\
\phi_k(t) \geq 0 \quad : \quad \text{for all} \quad t \in \mathbb{T} \\
\phi_k(t) = 0 \quad : \quad \text{for all} \quad t \geq \pi.
\]

Let \( \{\epsilon_k\}_{k=1}^{\infty} \) be a strictly monotonically decreasing sequence of positive real numbers with the property \( \lim_{k \to \infty} \epsilon_k \log k = \infty \) and set \( \delta_k := \epsilon_k - \epsilon_{k+1} \) for all \( k = 1, 2, \ldots, \). By this definition, \( \delta_k > 0 \) for all \( k \in \mathbb{N} \) and \( \sum_{k=1}^{\infty} \delta_k = \epsilon_1 - \epsilon_{\infty+1} \). Since \( \epsilon_n \to 0 \) as \( N \to \infty \), we have \( \sum_{k=1}^{\infty} \delta_k = \epsilon_n \). Now we define the function

\[
f_\epsilon(t) = \sum_{i=0}^{\infty} \delta_i \phi_i(t), \quad t \in \mathbb{T}.
\]
By the properties of $\phi_1$ and the triangle inequality, we have

$$\|f_*\|_\infty \leq \sum_{i=3}^{\infty} |\delta_i| \|\phi_i\|_\infty = \varepsilon_3 < \infty$$

and similarly $\|\tilde{f}_*\|_\infty \leq C_3 \varepsilon_3$. This shows that $f_* \in B$.

For an arbitrary $N \in \mathbb{N}$, we consider now $H_N^T f_*$ at the point $t = \pi$. This gives

$$\left(H_N^T f_*(\pi) = \sum_{k=0}^{L_N} f(k \frac{2\pi}{N}) \tilde{F}_N(\pi - k \frac{2\pi}{N}) \right)$$

$$\geq \sum_{i=3}^{\infty} \delta_i \left[ \sum_{k=0}^{L_N} \phi_i(k \frac{2\pi}{N}) \tilde{F}_N(\pi - k \frac{2\pi}{N}) \right]$$

where for the last inequality, it was used that $\delta_i > 0$ and that the expression in the brackets is positive (by Property (i) of Lemma 5 and because $\phi_i(\tau) \geq 0$ for all $\tau \in \mathbb{T}$). By the construction of $\phi_i$, it follows that $\phi_i(\lambda) = 1$ for all $\lambda \in T_M$ with $M \leq i$. Therefore, one obtains

$$\left(H_N^T f_*(\pi) \geq \sum_{i=3}^{\infty} \delta_i \left[ \sum_{k=0}^{L_N} \phi_i(k \frac{2\pi}{N}) \tilde{F}_N(\pi - k \frac{2\pi}{N}) \right] \geq \left( \delta_i \log N - C_0 \right) \varepsilon_N \right)$$

where the last inequality follows from Property (ii) of Lemma 5. Consequently, $\left(H_N^T f_*(\pi) \to \infty \right)$ as $N \to \infty$.

V. Approximations with Finite Search Horizon

Section IV showed strong divergence for the approximation method $\{H_N^T f_\}$. We believe that a similar result holds for all sequences $\{H_N\}_{N \in \mathbb{N}}$ with properties (A), (B), and (C). Up to now we can only state a weaker result.

**Theorem 7**: Let $\{H_N\}_{N \in \mathbb{N}}$ be a sequence of linear operators with properties (A), (B), and (C). Then there exists a function $f \in B$ with the following property:

To all numbers $M, N_0 \in \mathbb{N}$ and for every $\delta \in (0, 1)$ there exist two natural numbers $N^{(1)} = N^{(1)}(M, N_0, \delta)$ and $N^{(2)} = N^{(2)}(M, N_0, \delta)$ with

$$N^{(2)} > N^{(1)} \geq N_0 \quad \text{and} \quad \left[ N^{(2)} - N^{(1)} \right] / N^{(2)} > 1 - \delta$$

such that $\|H_N f\|_\infty > M$ for all $N \in [N^{(1)}, N^{(2)}]$.

For a given approximation sequence $\{H_N\}_{N \in \mathbb{N}}$, Theorem 7 shows that there exist functions $f \in B$ and arbitrarily long intervals $[N^{(1)}, N^{(2)}]$ on $\mathbb{N}$ such that $\|H_N f\|_\infty$ is arbitrarily large for every $N \in [N^{(1)}, N^{(2)}]$.

More precisely, let $f \in B$ be a function as in Theorem 7, and set $D(M, f) = \{N \in \mathbb{N} : \|H_N f\|_\infty > M\}$, then it follows from Theorem 7 that

$$\limsup_{K \to \infty} \frac{1}{K} |D(M, f) \cap [1, 2, \ldots, K]| = 1$$

However, we emphasize that Theorem 7 does not imply that $\{H_N\}_{N \in \mathbb{N}}$ diverges strongly on $B$. It only shows that all adaptive approximation methods with a finite search horizon diverge on $B$.

**Proof**: We choose a sequence $\{H_N\}_{N \in \mathbb{N}}$ with properties (A), (B), and (C). By Theorem 4 there exists an $f_0 \in C(T)$ with $\|f_0\|_\infty = 1$ such that $\lim_{N \to \infty} \|H_N f_0\|_\infty = \infty$. Now, we define recursively a sequence $\{\phi_k\}_{k=1}^\infty$ in $B$ whose limit will be the function $f \in B$ with the properties claimed by theorem.

**Step 1**: Let $N^{(1)}_1 \in \mathbb{N}$ be the smallest number such that $\|H_N f_0\|_\infty > 1$ for all $N \geq N^{(1)}_1$, and let $N^{(2)}_1 \in \mathbb{N}$ be the smallest number such that $\left[ N^{(2)}_1 - N^{(1)}_1 \right] / N^{(2)}_1 > 1/2$.

Property (A) of $\{H_N\}_{N \in \mathbb{N}}$ shows that to every $N \in \mathbb{N}$ there exists a finite sampling set $A_N$. We set

$$\Pi_1 = \bigcup_{N \in \mathbb{N}} \left[ N^{(1)}_1, N^{(2)}_1 \right] A_N$$

and apply Lemma 2 to define a function $\psi_1 \in B$ with the following properties

$$\begin{align*}
(i) & \quad \psi_1(\lambda) = f_0(\lambda) \quad \text{for all } \lambda \in \Pi_1, \\
(ii) & \quad \|\psi_1\|_\infty \leq 1 \quad \text{and} \quad (iii) \quad \|\tilde{\psi}_1\|_\infty \leq C_3, \\
(iv) & \quad \psi_1 \in C(\mathbb{T}), \end{align*}$$

Because of (iii) and (iv) and by Property (B) of $\{H_N\}_{N \in \mathbb{N}}$, there is an $N^{(3)}_1 > N^{(2)}_1$ such that $\|H_N \psi_1 - H_N f_0\|_\infty \leq 1$ for all $N \geq N^{(3)}_1$.

By the triangular inequality, this implies

$$\|H_N \psi_1\|_\infty \leq 1 + C_3 \quad \text{for all } N \geq N^{(3)}_1.$$

Now we set $\phi_1 = \psi_1$.

**Step k**: Assume that for $k \in \mathbb{N}$ arbitrary, we already defined the numbers $N^{(1)}_k, N^{(2)}_k, N^{(3)}_k$, the set $\Pi_k$, and the functions $\psi_k, \phi_k \in B$.

**Step k + 1**: Let $N^{(1)}_{k+1} \in \mathbb{N}$ be the smallest number such that $N^{(1)}_{k+1} > N^{(3)}_k$ and such that

$$\max_{t \in \mathbb{T}} \left| \sum_{n=1}^{M_N} f_0(\lambda, N) h_{n, N}(t) \right| > (k + 1)^3,$$

for all $N \geq N^{(1)}_{k+1}$. Moreover, let $N^{(2)}_{k+1} \in \mathbb{N}$ be the smallest number such that

$$\left[ N^{(2)}_{k+1} - N^{(1)}_{k+1} \right] / N^{(2)}_{k+1} > k / (k + 1)$$

and set $\Pi_{k+1} = \bigcup_{N \in \mathbb{N}} \left[ N^{(1)}_{k+1}, N^{(2)}_{k+1} \right] A_N$. By using Lemma 2, we define $\psi_{k+1} \in B$ with the following properties

$$\begin{align*}
(i) & \quad \psi_{k+1}(\lambda) = \begin{cases} f_0(\lambda) : & \lambda \in \Pi_{k+1} \Pi_k, \\
0 & \lambda \in \Pi_k, \\
\end{cases} \\
(ii) & \quad \|\psi_{k+1}\|_\infty \leq 1, \quad \text{and} \quad (iii) \quad \|\tilde{\psi}_{k+1}\|_\infty \leq C_3, \\
(iv) & \quad \psi_{k+1} \in C(\mathbb{T}). \end{align*}$$

Let $N^{(3)}_{k+1} > N^{(2)}_{k+1}$ be the smallest integer such that

$$\|H_N \psi_{k+1}\|_\infty \leq 1 + C_3 \quad \text{for all } N \geq N^{(3)}_{k+1}. \quad (8)$$

Note that by this definition of $\psi_{k+1}$, (7) implies that

$$\|H_N \psi_{k+1}\|_\infty \geq (k + 1)^3 \quad \text{for all } N \geq N^{(1)}_{k+1}. \quad (9)$$
Therewith, we define the function
\[ \phi_{k+1}(t) = \phi_k(t) + \frac{1}{(k+1)^r} \psi_{k+1}(t), \quad t \in \mathbb{T}. \]

The previous construction defined a sequence \( \{\phi_N\}_{N \in \mathbb{N}} \) in \( B \) where \( \phi_N \) and its Hilbert transform \( \tilde{\phi}_N \) are given by
\[ \phi_N = \sum_{n=1}^{N} \frac{1}{n^r} \psi_n \quad \text{and} \quad \tilde{\phi}_N = \sum_{n=1}^{N} \frac{1}{n^r} \tilde{\psi}_n, \]
respectively. By the properties of the functions \( \psi_k \), we have
\[ \|\phi_N\|_{\infty} \leq \frac{\pi^2}{4} \quad \text{and} \quad \|\tilde{\phi}_N\|_{\infty} \leq \frac{\pi^2}{4} C_3 \]
for all \( N \in \mathbb{N} \). This shows that \( \phi_N \) converges uniformly to a function \( f \in B \) as \( N \to \infty \) with \( \|f\|_B \leq \frac{\pi^2}{4} \max(1, C_3) \).

Now, let \( r \in \mathbb{N} \) be arbitrary and \( N \in [N^{(1)}, N^{(2)}] \) with the sampling set \( \Lambda_N \). By the definition of \( f \), we have
\[ \|H_N f\|_{\infty} = \left\| \sum_{n=1}^{N} \frac{H_N \psi_n}{n^r} + \sum_{n=r+1}^{\infty} \frac{H_N \psi_n}{n^2} \right\|_{\infty}. \]
By the construction of \( \psi_n \), we know that \( \psi_n(\lambda) = 0 \) for all \( \lambda \in \Lambda_K \) with \( K \in [N^{(1)}, N^{(2)}] \) and \( l < n \). So in particular \( \psi_n(\lambda) = 0 \) for all \( \lambda \in \Lambda_N \) for all \( n > r \). By Property (A) of \( \{H_N\} \), it follows that \( H_N \psi_n = 0 \) for all \( n > r \) such that
\[ \|H_N f\|_{\infty} \geq \frac{1}{\pi^2} \|H_N \psi_r\|_{\infty} - \sum_{n=1}^{r-1} \frac{1}{\pi^2} \|H_N \psi_n\|_{\infty}. \]
Now, (9) implies that \( \|H_N \psi_r\|_{\infty} \geq r^3 \) for \( N \geq N^{(1)} \), and (8) implies that \( \|H_N \psi_n\|_{\infty} \leq 1 + C_3 \) for all \( n \) for which \( N^{(3)} \leq N \leq N^{(2)} \), i.e. for all \( n < r \). Therewith, we have
\[ \|H_N f\|_{\infty} \geq r - (1 + C_3) \frac{\pi^2}{4} \lim_{N \to \infty} N \]
for all \( N \in [N^{(1)}, N^{(2)}] \), and by our construction \( [N^{(2)} - N^{(1)}]/N_r \to 1 - 1/r \).

Finally, we choose \( M, \delta, \) and \( N_0 \) arbitrarily and set \( r \) such that \( r \geq 1 > \delta, N^{(1)} > N_0 \) and \( r > M + (1 + C_3) \frac{\pi^2}{4} \). Then the above constructed \( f \in B \) and the two numbers \( N^{(1)} \) and \( N^{(2)} \) satisfy the statements of the theorem.

VI. The Size of the Divergence Set

Theorem 7 shows that there are functions \( f \in B \) such that any reasonable linear approximation \( H_N f \) of its Hilbert transform gets arbitrarily large on arbitrarily long intervals of the approximation index \( N \). The next theorem investigates the size of the corresponding divergence set.

**Theorem 8:** Let \( \{H_N\}_{N \in \mathbb{N}} \) be an operator sequence with properties (A), (B), and (C), and denote by \( D_H \) the set of all \( f \) in \( B \) for which the following holds: For arbitrary numbers \( M_0, N_0 \in \mathbb{N} \) and \( \delta \in (0, 1) \) there exist natural numbers \( N^{(1)} = N^{(1)}(M_0, \delta) \geq N_0 \) and \( N^{(2)} = N^{(2)}(M_0, \delta) > N^{(1)} \) with \( [N^{(2)} - N^{(1)}]/N^{(2)} > 1 - \delta \) such that
\[ \|H_N f\|_{\infty} > M \quad \text{for all} \quad N \in [N^{(1)}, N^{(2)}]. \]

The set \( D_H \) is a residual set in \( B \).

*Remark:* We know [5], [14] (cf. also the discussion in Sec. I) that the set for which strong divergence occurs is a meager set in \( B \). However, Theorem 8 shows that if we only allow for approximation methods with a finite search horizon, then divergence occurs on a residual set. So adaptive methods with finite search horizon give no improvement compared to non-adaptive methods, as far as the size of the divergence set is concerned.

**Sketch of proof:** The proof follows the general ideas already used in [3]. We begin by defining
\[ D(M; k) = \bigcap_{N_0=1}^{\infty} \bigcup_{N=N_0}^{N^{(1)}} \left\{ f \in B : \|H_N f\|_{\infty} > M \right\}. \]
In words, \( f \in B \) belongs to \( D(M; k) \) if and only if to every \( N_0 \in \mathbb{N} \) there is an \( N^{(1)} > N_0 \) such that \( \|H_N f\|_{\infty} > M \) for all \( N \in [N^{(1)}, N^{(2)}] \). In a first step, one shows that
\[ D_H = \bigcap_{k=1}^{\infty} \bigcap_{M=1}^{\infty} D(M; k). \]
Then one has to show that (10) is a residual set in \( B \). To this end, one considers the sets
\[ D_{00}(N_0, M, k) = \bigcup_{N=N_0}^{N^{(1)}} D(M, N^{(1)}, k). \]
and shows that these sets are open and dense in \( B \). Consequently, every \( D_{00}(N_0, M, k) \) is a residual set. By (10) and from the definition of \( D(M; k) \), we know that
\[ D_H = \bigcap_{k=1}^{\infty} \bigcap_{M=1}^{\infty} D_{00}(N_0, M, k). \]
Since the countable intersection of residual sets is again a residual set, one obtains that \( D_H \) is a residual set.

**References**


