Time-frequency representations for nonlinear frequency scales - Coorbit spaces and discretization

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***Abstract*—The fixed time-frequency resolution of the short-time Fourier transform has often been considered a major drawback. In this contribution we review recent results on a class of time-frequency transforms that adapt to a large class of frequency scales in the same sense that wavelet transforms are adapted to a logarithmic scale. In particular, we show that each transform in this class of *warped time-frequency representations* is a tight continuous frame satisfying orthogonality relations similar to Moyal's formula. Moreover, they satisfy the prerequisites of generalized coorbit theory, giving rise to coorbit spaces and associated discrete representations, i.e. atomic decompositions and Banach frames.**

I. Introduction

In this contribution, we introduce the notion of warped time-frequency transforms, a class of integral transforms representing functions in phase space with respect to nonlinear frequency scales. In particular, we show that warped time/frequency representations

(a) form tight continuous frames and thus are invertible.
(b) give rise to classes of generalized coorbit spaces, i.e. nested Banach spaces of functions with a certain localization in the associated phase space.
(c) are stable under a sampling operation, yielding atomic decompositions and Banach frames.

The desire for time-frequency transforms providing adapted time-frequency resolution has sparked a wealth of research and the construction of various systems with vastly different properties. The most prominent such systems are those in the extended *wavelet* family, e.g. the continuous wavelet [1], shearlet [2] or curvelet [3] transforms, all of which are adapted to a logarithmic frequency scale. Other important examples include α-transforms [4]–[7], a parametrized family of transforms adapted to frequency scales between linear and logarithmic, or the more abstract continuous and discrete nonstationary Gabor transforms [8], [9] and generalized translation-invariant systems [10] that provide a general framework for translation- or modulation-invariant, continuously indexed systems. For a more extensive list, see e.g. [11].

It is highly desirable that a time-frequency transform is invertible and possesses a norm-equivalence property, i.e. it forms a continuous frame [12], [13]. Moreover, we would like to retain these properties for appropriately sampled transform coefficients. In other words, we would like to obtain (discrete) frames [14], [15], or more generally atomic decompositions and Banach frames [16], [17]. Under certain conditions on the transform, such results are obtained via coorbit theory [18], [19] and its various generalizations, see e.g. [2], [20]–[22]. Coorbit theory also provides the appropriate Banach space families to investigate these questions in.

Herein, we introduce a novel family of time-frequency representations adapted to nonlinear frequency scales. Uniquely determined by the choice of a single prototype atom and a warping function that determines the desired frequency scale, our construction provides a family of time-frequency atoms with uniform frequency resolution *when observed on the chosen frequency scale*. For particular choices of the warping function, we recover the continuous short-time Fourier and wavelet transforms. Hence, the proposed *warped time-frequency representations* can be considered a unifying framework for a larger class of time-frequency systems that form tight continuous frames and a highly structured special case of the systems considered in [8], [10].

Note that the results in this contribution are compiled from [11], where a detailed study of warped time-frequency representations and proofs of the results presented here can be found.

The considerable notation and prerequisites for establishing these results are presented in Sections II and III. In Section IV we introduce warped time-frequency representations and discuss the tight frame property. We also provide sufficient conditions for the construction of coorbit spaces with respect to a warped time-frequency representation. Finally, Section V is concerned with discretization of warped time-frequency systems. The results therein enable the construction of atomic decompositions and Banach frames of warped time-frequency systems.

II. Notation and preliminaries

We write $\hat{f}(\xi) := \mathcal{F}f(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi it\xi}\ dt$, for the Fourier transform on $\mathbf{L}^1(\mathbb{R})$ and its unitary extension to $\mathbf{L}^2(\mathbb{R})$, denoting its inverse by $\check{f} := \mathcal{F}^{-1}f$. Further, we require

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the *translation operator* defined by $\mathbf{T}_x f = f(\cdot - x)$. Note that, for a Banach space B, we denote its anti-dual, i.e. the space of all continuous, conjugate-linear functionals on B, by B^*. The Landau notation $\mathcal{O}(f)$ denotes all functions that do not grow faster than f.

Let $\mathcal{H}$ be a separable Hilbert space and (X, μ) a locally compact, σ-compact Hausdorff space with positive Radon measure μ on X.

A collection $\Psi = \{\psi_x\}_{x \in X}$ of functions $\psi_x \in \mathcal{H}$ is called a *continuous frame*, if there are $0 < A \leqslant B < \infty$, such that

$$A\|f\|_{\mathcal{H}}^2 \leqslant \int_X |\langle f, \psi_x \rangle|^2 d\mu(x) \leqslant B\|f\|_{\mathcal{H}}^2,$$

for all $f \in \mathcal{H}$, and the map $x \mapsto \psi_x$ is weakly continuous. A frame is *tight*, if $A = B$. The *frame operator* defined (in the weak sense) by

$$\mathbf{S}_\Psi \ : \ \mathcal{H} \to \mathcal{H}, \ \mathbf{S}_\Psi f := \int_X \langle f, \psi_x \rangle \psi_x d\mu(x),$$

is bounded, positive and boundedly invertible [12].

The application of a *kernel* $K : X \times X \to \mathbb{C}$ to a function G on X is given by

$$K(G)(x) := \int_X K(x,y) G(y) \ d\mu(y).$$

For the sake of brevity, we will assume from now on that $\Psi := \{\psi_x\}_{x \in X} \subset \mathcal{H}$ is a tight frame, i.e. $\mathbf{S}_\Psi f = Af$ for all $f \in \mathcal{H}$. Define the following transform associated to Ψ,

$$V_\Psi \ : \ \mathcal{H} \to L^2(X, \mu), \quad V_\Psi(x) := \langle f, \psi_x \rangle. \tag{1}$$

Furthermore, let $\mathcal{A}_1$ be the Banach algebra of all kernels $K : X \times X \to \mathbb{C}$, such that the norm

$$\|K\|_{\mathcal{A}_1} := \max \left\{ \operatorname*{ess\,sup}_{x \in X} \int_X |K(x,y)| \ d\mu(y), \right.$$
$$\left. \operatorname*{ess\,sup}_{y \in X} \int_X |K(x,y)| \ d\mu(x) \right\}$$

is finite, with the algebra multiplication $K_1 \cdot K_2(x,y) = \int_X K_1(x,z) K_2(z,y) \ d\mu(z)$.

For any weight function $m : X \times X \to \mathbb{C}$ that satisfies

$$1 \leqslant m(x,y) \leqslant m(x,z) m(z,y),$$
$$m(x,y) = m(y,x) \ \text{ and } \ m(x,x) \leqslant C,$$

for some $C > 0$, we say m is *admissible*, and all $x, y, z \in X$, $\mathcal{A}_m$ is the subalgebra of kernels $K : X \times X \to \mathbb{C}$, such that

$$\|K\|_{\mathcal{A}_m} := \|Km\|_{\mathcal{A}_1} < \infty.$$

From [20], we have the following results.

Theorem 1. *Let m be an admissible weight function, fix $z \in X$ and define $v := m(\cdot, z)$. If $\Psi \subset X$ is a continuous tight frame and the kernel $K_\Psi : X \times X \to \mathbb{C}$, given by*

$$K_\Psi(x,y) := A^{-1} \langle \psi_y, \psi_x \rangle \text{ for all } x, y \in X, \tag{2}$$

is contained in $\mathcal{A}_m$, then

$$\mathcal{H}_v^1 := \{ f \in \mathcal{H} \ : \ V_\Psi f \in \mathbf{L}_v^1 \}, \tag{3}$$

with the norm $\|f\|_{\mathcal{H}_v^1} := \|V_\Psi f\|_{\mathbf{L}_v^1}$, is the minimal Banach space *containing all the frame elements ψ_x and satisfying $\|\psi_x\|_B \leqslant Cv(x)$ for some $C > 0$. Furthermore, $\mathcal{H}_v^1$ is independent of the particular choice of $z \in X$ and $\|V_\Psi f\|_{\mathbf{L}^\infty_{1/v}}$ defines an equivalent norm on the anti-dual $(\mathcal{H}_v^1)^*$ of $\mathcal{H}_v^1$.*

The result above enables the extension of V_Ψ to the distribution space $(\mathcal{H}_v^1)^*$ by means of

$$V_\Psi f(x) := \langle f, \psi_x \rangle = f(\psi_x), \tag{4}$$

for all $x \in X, f \in (\mathcal{H}_v^1)^*$. For an exhaustive list of the attractive properties of $\mathcal{H}_v^1$, see [20].

If a *solid* Banach space Y satisfies

$$\mathcal{A}_m(Y) \subset Y \text{ and } \|K(F)\|_Y \leqslant \|K\|_{\mathcal{A}_m} \|F\|_Y, \tag{5}$$

for all $K \in \mathcal{A}_m, \ F \in Y$ then, provided $K_\Psi \in \mathcal{A}_m$, we can define the coorbit of Y with respect to Ψ as

$$\mathrm{Co}(\Psi, Y) := \{ f \in (\mathcal{H}_v^1)^* \ : \ V_\Psi f \in Y \}, \tag{6}$$

with natural norms $\|f\|_{\mathrm{Co}(\Psi,Y)} := \|V_\Psi f\|_Y$. When the association of the coorbit to a continuous frame is clear, we write $\mathrm{Co}Y := \mathrm{Co}(\Psi, Y)$.

By [20, Proposition 3.7], we have the following properties of $(\mathrm{Co}Y, \| \cdot \|_{\mathrm{Co}Y})$:

(a) $\mathrm{Co}Y$ is a Banach space,
(b) $G \in Y$, $G = K_\Psi(G) \Leftrightarrow G = V_\Psi f$ for some $f \in \mathrm{Co}Y$ and
(c) $V_\Psi : \mathrm{Co}Y \to Y$ is an isometry on the closed subspace $K_\Psi(Y)$ of Y.

III. Discretization and the $(\mathcal{U}, \Gamma)-$oscillation

We are interested in when a countable subfamily of $\Psi = \{\psi_i\}_{i \in I}$ allows the expansion of any function in $f \in \mathrm{Co}Y$ and/or the coefficients $(\langle \cdot, \psi_i \rangle)_{i \in I}$ uniquely describe every function $f \in \mathrm{Co}Y$. Thus, we are looking for *atomic decompositions* and *Banach frames* of $\mathrm{Co}Y$.

Essentially, $\{\psi_i\}_{i \in I}$ is an atomic decomposition of $\mathrm{Co}Y$, if there are linear bounded functionals $(\lambda_i)_{i \in I} \subset \mathrm{Co}Y^*$, such that $f = \sum_{i \in I} \lambda_i(f) \psi_i$ for all $f \in \mathrm{Co}Y$ and a Banach frame for $\mathrm{Co}Y$, if there is a linear bounded operator Ω such that $\Omega(\langle f, \psi_i \rangle)_{i \in I} = f$ for all $f \in \mathrm{Co}Y$. For a formal definition and the necessary technical details, see [17], [20].

Before we can proceed, we need the notion of *moderate, admissible coverings* of X.

Definition 1. A family $\mathcal{U} = \{U_i\}_{i \in I}$ for some countable index set I is called *admissible covering* of X, if the following hold. Every U_i is relatively compact with non-void interior, $X = \bigcup_{i \in I} U_i$ and $\sup_{i \in I} \#\{j \in I \ : \ U_i \cap U_j \neq \varnothing\} \leqslant N < \infty$ for some $N > 0$.

An admissible covering is *moderate*, if $0 < D \leqslant \mu(U_i)$ for all $i \in I$ and there is a constant $\tilde{C}$ with

$$\mu(U_i) \leqslant \tilde{C} \mu(U_j), \text{ for all } i, j \in I \text{ s.t. } U_i \cap U_j \neq \varnothing.$$

From now on we require that there exists a moderate, admissible covering $\mathcal{U} = (U_i)_{i\in I}$ of X and a constant $C_{m,\mathcal{U}}$ such that

$$\sup_{x,y\in U_i} m(x,y) \leqslant C_{m,\mathcal{U}} \text{ for all } i \in I. \tag{7}$$

Definition 2. A frame $\mathcal{F}$ is said to possess property $D[\delta, m]$ if there exists a moderate admissible covering $\mathcal{U}$ of X and a phase function $\Gamma : X \times X \to \mathbb{C}$ with $|\Gamma| = 1$ such that (7) holds and the kernel

$$\operatorname{osc}_{\mathcal{U},\Gamma}(x,y) := A^{-1} \sup_{z\in Q_y} |\langle \psi_x, \psi_y - \Gamma(y,z)\psi_z\rangle|,$$

$Q_y := \bigcup_{i,y\in U_i} U_i$, satisfies $\|\operatorname{osc}_{\mathcal{U},\Gamma}\|_{\mathcal{A}_m} < \delta$.

Note that, compared to the definition in [20], we allow for an additional phase factor Γ. This modification is required for the main result presented in Section V, see also [11] for an extended discussion on the implications of this additional freedom.

All the results in [20] are easily extended to this setting, only requiring a number of trivial modifications to the derivations presented in [20, Section 5], see also [23] where we provide an extended and annotated variant of [20, Section 5] including the necessary changes to accommodate the modified Definition 2.

In particular, we obtain the following result, the crucial ingredient for the construction of atomic decompositions and Banach frames.

Theorem 2. *Assume that m is an admissible weight. Suppose the frame $\Psi = \{\psi_x\}_{x\in X}$ possesses property $D[\delta, m]$ for some $\delta > 0$ and let $\mathcal{U}^\delta$ denote a corresponding covering of X such that*

$$\delta(C_\Psi + \max\{C_{m,\mathcal{U}^\delta} C_\Psi, C_\Psi + \delta\}) \leqslant 1 \tag{8}$$

where $C_{m,\mathcal{U}^\delta}$ is the constant in (7) and $C_\Psi = \|K_\Psi\|_{\mathcal{A}_m}$. Choose points $(x_i)_{i\in I} \subset X$ such that $x_i \in U_i$. Moreover assume that $(Y, \|\cdot|Y\|)$ is a solid Banach space fulfilling (5).

Then $\Psi_d := \{\psi_{x_i}\}_{i\in I} \subset \mathcal{H}^1_v$ is both an atomic decomposition of CoY and a Banach frame for CoY.

IV. Warped time-frequency representations

Now we introduce the notion of a warping function and warped time-frequency systems. The remainder of this contribution is concerned with presenting the attractive properties of these systems in the Hilbert space and coorbit space settings. From now on, we assume $X = D \times \mathbb{R}$, where $D \in \{\mathbb{R}, \mathbb{R}^+\}$ and $\mathcal{H} = \mathcal{F}^{-1}(\mathbf{L}^2(D))$. As prototypical examples for the solid Banach space Y, we can consider a weighted, mixed-norm space from the family $\mathbf{L}^{p,q}_w(X)$, for $1 \leqslant p, q \leqslant \infty$ and a continuous, nonnegative weight function $w : X \mapsto \mathbb{R}$. These spaces consist of all Lebesgue measurable functions, such that the norm

$$\|G\|_{\mathbf{L}^{p,q}_w} := \left(\int_R \left(\int_R w(x,\xi)^p |G(x,\xi)|^p \, dx \right)^{q/p} d\xi \right)^{1/q}$$

is finite.

Definition 3. Let $D \in \{\mathbb{R}, \mathbb{R}^+\}$. A bijective function $F : D \to \mathbb{R}$ is called *warping function*, if $F \in \mathcal{C}^1(D)$ with $F' > 0$, $|t_0| < |t_1| \Rightarrow F'(t_1) < F'(t_0)$ and the associated weight function

$$w(t) = \left(F^{-1}\right)'(t),$$

is v-moderate for some submultiplicative weight v. If $D = \mathbb{R}$, we additionally require F to be odd.

If $\theta \in \mathbf{L}^2_w(\mathbb{R})$, then the *warped time-frequency system* with respect to θ and F is defined by $\mathcal{G}(\theta, F) := \{g_{x,\xi}\}$, where

$$g_{x,\xi} := T_\xi \check{g_x}, \quad g_x := \sqrt{F'(x)}(\mathbf{T}_{F(x)}\theta) \circ F,$$

for all $x \in D,\ \xi \in \mathbb{R}$.

Given a warping function F and the prototype $\theta \in \mathbf{L}^2_w(\mathbb{R})$, the respective integral (frame) transform is given by $V_{\theta,F} f(x,\xi) := V_{\mathcal{G}(\theta,F)} f(x,\xi) = \langle f, g_{x,\xi}\rangle$, for all $f \in \mathcal{F}^{-1}(\mathbf{L}^2(D))$. The associated *phase space* is $D \times \mathbb{R}$. Clearly, $\mathcal{G}(\theta, F) \subset \mathcal{F}^{-1}\mathbf{L}^2(D)$.

It is easy to see that $V_{\theta,F} f \in \mathbf{L}^\infty(D\times\mathbb{R})$. However, $F \in \mathcal{C}^1$ and w being v-moderate actually imply $V_{\theta,F} f \in \mathcal{C}(D \times \mathbb{R})$. Indeed, $V_{\cdot,F}$ also possesses the following norm-equivalence property that implies invertibility.

Theorem 3. *Let F be a warping function and $\theta_1, \theta_2 \in \mathbf{L}^2_w$. Furthermore, assume that θ_1 and θ_2 fulfill the admissibility condition*

$$|\langle \theta_1, \theta_2\rangle| < \infty. \tag{9}$$

Then the following holds for all $f_1, f_2 \in \mathcal{F}^{-1}(\mathbf{L}^2(D))$:

$$\int_D \int_{\mathbb{R}} V_{\theta_1,F} f_1(x,\xi) \overline{V_{\theta_2,F} f_2(x,\xi)} \, d\xi dx = \langle f_1, f_2\rangle\langle \theta_2, \theta_1\rangle.$$

In particular, if $\theta \in \mathbf{L}^2_w$ is normalized in the (unweighted) $\mathbf{L}^2$ sense, then $\|V_{\theta,F} f\|_{\mathbf{L}^2} = \|f\|_{\mathbf{L}^2}$.

Corollary 4. *Given a warping function F and some nonzero $\theta \in \mathbf{L}^2_w \cap \mathbf{L}^2$. Then any $f \in \mathcal{F}^{-1}(\mathbf{L}^2(D))$ can be reconstructed from $V_{\theta,F} f$ by*

$$f = \frac{1}{\|\theta\|_{\mathbf{L}^2}} \int_D \int_{\mathbb{R}} V_{\theta,F} f(x,\xi) g_{x,\xi} \, d\xi dx.$$

The equation holds in the weak sense.

The theorem above also implies the tight frame property for the warped system $\mathcal{G}(\theta, F)$. The following result provides sufficient conditions for $\mathcal{G}(\theta, F)$ such that the associated kernel $K_{\mathcal{G}(\theta,F)}$ is contained in $\mathcal{A}_m$ for weights m depending on the frequency variables $x, y \in D$ only. This enables the construction of coorbit spaces for warped time-frequency representations by means of Theorem 1.

Theorem 5. *Let $F : D \mapsto \mathbb{R}$ be a warping function with $w = (F^{-1})' \in \mathcal{C}^1(\mathbb{R})$, such that for all $x, y \in \mathbb{R}$:*

$$\frac{w(x+y)}{w(x)w(y)} \leqslant C < \infty \text{ and } \left|\frac{w'}{w}\right| \leqslant D_1 < \infty.$$

Furthermore, let $m_1 : D \to \mathbb{R}$ such that $m_1 \circ F^{-1}$ is v_1-

moderate, for a symmetric, submultiplicative weight function v_1 *and define* $m(x,y,\xi,\omega) = \max\left\{\frac{m_1(x)}{m_1(y)}, \frac{m_1(y)}{m_1(x)}\right\}$. *Then*

$$K_{\mathcal{G}(\theta,F)} \in \mathcal{A}_m, \quad \text{for all } \theta \in \mathcal{C}_c^\infty.$$

If furthermore $w, v_1 \in \mathcal{O}((1+|\cdot|)^p)$ *for some* $p \in \mathbb{R}^+$, *then*

$$K_{\mathcal{G}(\theta,F)} \in \mathcal{A}_m, \quad \text{for all } \theta \in \mathcal{S}.$$

Example 1. Choose $F = \log$, with $D = \mathbb{R}^+$, to obtain a system of the form

$$\begin{aligned} g_x(t) &= x^{-1/2}\theta(\log(t) - \log(x)) \\ &= x^{-1/2}\theta(\log(t/x)) = x^{-1/2} g_{F^{-1}(0)}(t/x). \end{aligned}$$

This warping function therefore leads to g_x being a dilated version of g_1. Consequently, $\mathcal{G}(\theta, \log)$ is a continuous wavelet system. Note however, that our scales are reciprocal to the usual definition of wavelets, see [1], [24]. Since $F^{-1}(x) = e^x$ and $(e^x)^p = e^{px}$, we can, for polynomial weights m_1, find a submultiplicative weight v_1 such that $m_1 \circ F^{-1}$ is v_1-moderate. Therefore, we can invoke Theorem 5 with any test function $\theta \in \mathcal{C}_c^\infty$ and $Y = L^{p,q}_{m_1}(D \times \mathbb{R})$.

Example 2. The warping function

$$F_l(t) = \operatorname{sgn}(t)\ \left((|t|+1)^l - 1\right),\ l \in]0,1],$$

has the inverse $F_l^{-1}(t) = \operatorname{sgn}(t)\ \left((|t|+1)^{1/l} - 1\right)$. Hence, for any polynomial m_1, the composition $m_1 \circ F_l^{-1}$ is again polynomial and we can invoke Theorem 5 with any test function $\theta \in \mathcal{S}$. If on the other hand m_1 is subexponential with $m_1(x) \in \mathcal{O}(e^{|x|^l})$, then $m_1 \circ F_l^{-1}$ has at most exponential growth and is v_1-moderate for some submultiplicative weight v_1. Consequently, $\theta \in \mathcal{C}_c^\infty$ is sufficient for Theorem 5 to provide $K_{\theta,F} \in \mathcal{A}_m$.

Example 3. In [25] the authors construct a filter bank that is tailored to the Equivalent Rectangular Bandwidth (ERB) scale, see also [26], a frequency scale tailored to human auditory perception. Using the ERB warping function is given by

$$F(t) = \operatorname{sgn}(t)\ c_1 \log\left(1 + \frac{|t|}{c_2}\right),$$

where the constants are given by $c_1 = 9.265$ and $c_2 = 228.8$, we can easily construct a transform adapted to the ERB scale. Since F^{-1} has exponential growth, polynomial weights m_1 and prototypes $\theta \in \mathcal{C}_c^\infty$ yield $K_{\theta,F} \in \mathcal{A}_m$, by applying Theorem 5. A system constructed from this particular warping functions has possible applications in all analysis and processing tasks related to the human perception of sound.

Remark 1. In fact, Theorem 5 is just a special case of a more general result. In [11], we derive sufficient differentiability and decay conditions on w and θ, such that $K_{\theta,F} \in \mathcal{A}_m$. In particular, for $D = \mathbb{R}$ we allow $m(x,y,\xi,\omega) = \max\left\{\frac{m_1(x)m_2(\xi)}{m_1(y)m_2(\omega)}, \frac{m_1(y)m_2(\omega)}{m_1(x)m_2(\xi)}\right\}$, with a polynomial weight m_2.

V. Discrete warped time-frequency systems

We are now able to construct coorbit spaces with respect to warped time-frequency systems $\mathcal{G}(\theta, F)$. In this section, we aim to find discrete subfamilies that allow for stable expansion of all functions in those coorbit spaces (atomic decompositions) and/or reconstruction from the sampled transform coefficients (Banach frames).

To that end, we construct moderate, admissible coverings of $X = D \times \mathbb{R}$ for any admissible warping function F. These coverings are constructed in order to show that families of covers and a canonical choice of Γ exist, such that

$$\|\mathrm{osc}_{\mathcal{U}^\delta,\Gamma}\|_{\mathcal{A}_m} \overset{\delta\to 0}{\to} 0 \text{ and } C_{m,\mathcal{U}^\delta} \overset{\delta\to 0}{\to} C < \infty.$$

for sufficiently smooth, quickly decaying prototype θ.

Consequently, the discretization machinery provided in Section III can be put to work, providing atomic decompositions and Banach frames with respect to $\mathcal{G}(\theta, F)$ and the family of coverings $\mathcal{U}^\delta$, $\delta > 0$. Let us first define a prototypical family of coverings induced by the warping function.

Definition 4. Let F be a warping function that satisfies $w(x+y) \leqslant Cw(x)w(y)$, for all $x, y \in D$ and some $C > 0$. Define $\mathcal{U}_F^\delta = \{U_{l,k}^\delta\}_{l,k\in\mathbb{Z}}$, $\delta > 0$ by

$$U_{l,k}^\delta := I_{F,l}^\delta \times \left[\frac{\delta^2 k}{|I_{F,l}^\delta|}, \frac{\delta^2(k+1)}{|I_{F,l}^\delta|}\right], \tag{10}$$

where $I_{F,l}^\delta := \left[F^{-1}(\delta l), F^{-1}(\delta(l+1))\right]$. We call $\mathcal{U}_F^\delta$ the *induced δ-cover* with respect to F. For all $\delta > 0$, $\mathcal{U}_F^\delta$ is a moderate, admissible covering with $\mu(U_{l,k}^\delta) = \delta^2$, where μ is the Lebesgue measure.

Clearly, $\mathcal{U}_F^\delta$ can be replaced by any covering that is m-equivalent to $\mathcal{U}_F^\delta$, as per [20, Definition 5.3]. The induced δ-cover with respect to F prescribes a sort of density condition for the selection of a countable subfamily $(g_{x_i,\xi_i})_{i\in I}$, $(x_i, \xi_i) \in D \times \mathbb{R}$, of $\mathcal{G}(\theta, F)$. If this density is high enough, then the next result, combined with Theorem 2, guarantees that we obtain atomic decompositions and Banach frames.

Theorem 6. *Assume that the conditions of Theorem 5 hold. Let furthermore $\mathcal{U}_F^\delta$ be the induced δ-cover for F. There are constants*

$$C_{m,\mathcal{U}_F^\delta} \geqslant \sup_{k,l\in\mathbb{Z}} \sup_{(x,\xi),(y,\omega)\in U_{l,k}^\delta} m(x,y,\xi,\omega),$$

such that for sufficiently small δ it holds $C_{m,\mathcal{U}_F^\delta} < C < \infty$. Moreover, for all $\theta \in \mathcal{C}_c^\infty$ and $\epsilon > 0$

$$\exists\ \delta_\epsilon > 0 \text{ s.t. } \forall\ 0 < \delta \leqslant \delta_\epsilon\ :\ \|osc_{\mathcal{U}_F^\delta,\Gamma}\|_{\mathcal{A}_m} < \epsilon, \tag{11}$$

where $\Gamma(x,y,\xi,\omega) = e^{2\pi i(\omega-\xi)x}$. If furthermore $w, v_1 \in \mathcal{O}((1+|\cdot|)^p)$ for some $p \in \mathbb{R}^+$, then (11) *holds for all $\theta \in \mathcal{S}$ and $\epsilon > 0$.*

In fact, $osc_{\mathcal{U}_F^\delta,\Gamma} \in \mathcal{A}_m$, for all $\delta > 0$, under the conditions above.

Thus we can apply Theorem 2 for $\mathcal{G}(\theta, F)$ and $\mathcal{U}_F^\delta$ for sufficiently small $\delta > 0$, providing atomic decompositions and Banach frames of all $\mathrm{Co}Y$, uniformly over all Y with associated weight m. Note that the examples given in the previous section are valid choices to satisfy Theorem 6.

Remark 2. Similar to the previous section, Theorem 6 above is just a special case of a more general result, see [11], stating sufficient differentiability and decay conditions on w and θ, such that $\mathrm{osc}_{\mathcal{U}_F^\delta,\Gamma} \in \mathcal{A}_m$ and $\|\mathrm{osc}_{\mathcal{U}_F^\delta,\Gamma}\|_{\mathcal{A}_m} \to 0$ for $\delta \to 0$.

VI. Conclusion

Warped time-frequency representations provide a new way of constructing adapted representations for a large number of frequency scales, covering the linear (Gabor) and logarithmic (wavelet) scales and many more. Warped time-frequency systems form tight continuous frames and give rise a novel family of associated (generalized) coorbit spaces.

By a relaxation of previously known sufficient conditions for discretization results in generalized coorbit theory, we are able to obtain atomic decompositions and Banach frames by selecting appropriate countable subfamilies of warped time-frequency systems.

All in all, warped time-frequency representations provide a promising new alternative to more traditional representations adapted to nonlinear frequency scales, such as the α-transform.

An extended treatment of the properties of warped time-frequency representations as well as the proofs for the results presented here can be found in [11]. The discrete Hilbert frame properties of warped systems are discussed in [27].

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