Sampling and Recovery Using Multiquadrics

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Abstract—We survey recent results in the subject of interpolating bandlimited functions from their samples at both uniform and nonuniform sets via translates of a family of multiquadrics. Recovery of the original function is considered by means of a limiting process which changes a shape parameter associated with the multiquadric function.

We also discuss some ways in which approximation rates can be found as well as extensions of the results to interpolation schemes which use other radial basis functions.

I. CARDINAL INTERPOLATION

There has been, for some time, an overlap between sampling and recovery of bandlimited functions and radial basis function theory. The impetus for this interaction was an interesting idea of Isaac Schoenberg (see, for example, [16]), which was to replace the sinc function in the classical Whittaker-Kotelnikov-Shannon Sampling Theorem with another function which behaved similarly and yet decayed much more rapidly away from the origin than sinc, whose decay is $O(|x|^{-1})$. Precisely, one forms a so-called fundamental function, $L$, which has the property of the sinc function that $L(k) = \delta_{0,k}$ for every integer $k$ (where $\delta_{i,j}$ is the Kronecker delta). Schoenberg considered fundamental functions defined using cardinal splines. Given a fundamental function, one can consider the approximation of a bandlimited function $f$ given by

$$If(x) := \sum_{j \in Z} f(j)L(x-j), \quad x \in \mathbb{R}. \quad (1)$$

By the fundamental function condition on $L$, it is apparent that $If$ interpolates $f$ at the integer lattice. The tradeoff is generally this: one forgoes pointwise equality obtained by the classical sampling theorem in exchange for interpolation at the sample points and a series that converges uniformly, and moreover, one asks for $\|If-f\|_{L_2(\mathbb{R})}$ to be small. For example, the fundamental function associated with the cardinal splines satisfies $L(x) = O(e^{-|x|})$, $|x| \to \infty$.

Such fundamental functions and interpolants may be formed in any dimension to interpolate multivariate bandlimited functions. For the remainder of the current section, we will consider interpolation at the (multi) integer lattice in $\mathbb{R}^d$ for some dimension $d \in \mathbb{N}$.

A. General Multiquadrics

Schoenberg’s idea was taken up by Baxter [2], who considered the fundamental function formed from the Hardy multiquadric, $\phi(x) = \sqrt{\|x\|^2 + c^2}$, and subsequently by Riemenschneider and Sivakumar [14], who used the Gaussian kernel. Subsequent analysis has been done by the author and Ledford on cardinal interpolation using general multiquadrics [7], which are

$$\phi_{\alpha,c}(x) := (\|x\|^2 + c^2)^{\alpha}, \quad x \in \mathbb{R}^d, \quad (2)$$

where $\| \cdot \|$ denotes the Euclidean distance, and $c > 0$ and $\alpha \in \mathbb{R}$ are parameters. In the special case where $d = 1$ and $\alpha = -1$, the function $\phi_{-1,c}$ is called the Poisson kernel, and its Fourier transform is given by $\hat{\phi}_{-1,c}(\xi) = (2c)^{-1}e^{-c|\xi|}$, where the convention we have adopted is

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$  

For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, the generalized Fourier transform of the multiquadric can be expressed as a function:

$$\hat{\phi}_{\alpha,c}(\xi) = \frac{2^{1+\alpha}}{\Gamma(-\alpha)} \left( \frac{c}{\|\xi\|} \right)^{\alpha+\frac{d}{2}} K_{\alpha+rac{d}{2}}(c\|\xi\|), \quad \xi \in \mathbb{R}^d \setminus \{0\}, \quad (3)$$

where $\Gamma$ is the usual Gamma function, and $K_{\nu}$ is the modified Bessel function of the second kind, which may be defined by

$$K_{\nu}(r) := \frac{1}{2} \int_{0}^{\infty} e^{-r \cosh t} e^{\nu t}dt, \quad r > 0, \quad \nu \in \mathbb{R}. \quad (4)$$

The function $K_{\nu}$ decays exponentially away from the origin and has an algebraic pole at the origin, which for large enough negative values of $\alpha$ is canceled out by the polynomial in $\|\xi\|$ appearing in (3). These facts can be found, for example, in [17].

B. Fundamental Functions

The fundamental function associated with the general multiquadric is defined via its Fourier transform as follows:

$$\hat{L}_{\alpha,c}(\xi) := \frac{\hat{\phi}_{\alpha,c}(\xi)}{\sum_{k \in \mathbb{Z}^d} \phi_{\alpha,c}(\xi + 2\pi k)}, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \quad (5)$$

At the origin, we assign $\hat{L}_{\alpha,c}$ its limiting value, which exists, and in this case is 1. Owing to the exponential decay of the modified Bessel function, one can show by a standard periodization argument that $\hat{L}_{\alpha,c} \in L_1 \cap L_2(\mathbb{R}^d)$. It follows by defining $L_{\alpha,c}$ via the inverse Fourier integral and the use of the dominated convergence theorem that $L_{\alpha,c}$ is a fundamental function.

Theorem I.1. The function $L_{\alpha,c}$ is a fundamental function, and moreover has the form

$$L_{\alpha,c}(x) = \sum_{k \in \mathbb{Z}^d} c_k \phi_{\alpha,c}(x-k), \quad x \in \mathbb{R}^d, \quad (6)$$
where \( c_k \) are the Fourier coefficients of the \( 2\pi \)-periodic symbol \( \omega(\xi) = 1/\sum_{j \in \mathbb{Z}} \phi_{\alpha,c}(\xi + 2\pi j) \).

Sketch of Proof: To show (6), one proves that \( \omega \) is continuous, \( 2\pi \)-periodic, and has absolutely summable Fourier coefficients, and so is identified with its Fourier series \( \omega(\xi) = \sum_{k \in \mathbb{Z}} c_k e^{ikx} \). One must then justify the formal calculation

\[
L_{\alpha,c}(\xi) = \int_{\mathbb{R}^d} \omega(\xi) \hat{\phi}_{\alpha,c}(\xi) e^{i\xi \cdot x} d\xi = \sum_{k \in \mathbb{Z}} c_k \int_{\mathbb{R}^d} \phi_{\alpha,c}(\xi) e^{i\xi \cdot (x-k)} d\xi,
\]

which yields (6), at least in the case where \( \alpha \) is negative enough for the Fourier inversion formula to hold for the multiquadric. If this is not the case, then an argument similar to one used in [3] yields the result by using generalized Fourier transform arguments.

C. Recovery via Cardinal Interpolants

As previously suggested, let us now fix an \( \alpha \in \mathbb{R} \setminus \mathbb{N}_0 \), and let \( c > 0 \). Suppose \( f \) is a bandlimited function in the \( d \)-dimensional Paley Wiener space \( \mathbb{PW}^{[d]}_\pi := \{ f \in \mathbb{L}_2(\mathbb{R}^d) : f(\xi) = 0 \text{ a.e. outside } [-\pi, \pi]^d \} \). Then we form the cardinal multiquadric interpolant of \( f \) via

\[
I_{\alpha,c} f(x) = \sum_{k \in \mathbb{Z}^d} f(k) L_{\alpha,c}(x-k), \quad x \in \mathbb{R}^d.
\]  

(7)

By taking a limit as \( c \to \infty \), we may recover the function \( f \) both in \( \mathbb{L}_2(\mathbb{R}^d) \) and uniformly.

Theorem I.2. Let \( \alpha \in \mathbb{R} \setminus \mathbb{N}_0 \). If \( f \in \mathbb{PW}^{[d]}_\pi \), then

\[
\lim_{c \to \infty} \| I_{\alpha,c} f - f \|_{\mathbb{L}_2(\mathbb{R}^d)} = 0,
\]

and

\[
\lim_{c \to \infty} | I_{\alpha,c} f(x) - f(x) | = 0
\]

uniformly on \( \mathbb{R}^d \).

The preceding theorem is proved by using Plancherel’s identity and carrying out the analysis in the Fourier domain. An important observation, and indeed what drives the convergence for cardinal interpolation of bandlimited functions using cardinal splines, Gaussians, and multiquadrics is the fact that in each case, the fundamental functions converge pointwise to the sinc function. We state the equivalent fact for the Fourier transform of the fundamental function associated with the general multiquadric.

Proposition I.3. Let \( \chi_{[-\pi,\pi]^d} \) be the function that takes value \( 1 \) on \( [-\pi, \pi]^d \) and 0 elsewhere. Then for almost every \( \xi \in \mathbb{R}^d \),

\[
\lim_{c \to \infty} \hat{\phi}_{\alpha,c}(\xi) = \chi_{[-\pi,\pi]^d}(\xi).
\]

What this means is that, in a limiting sense, Theorem I.2 is simply recovering the classical sampling theorem. However, for each \( c \), the series in (7) converges uniformly, at least for a large range of \( \alpha \). It is for this reason that Schoenberg titled such processes summability methods.

For some good references on cardinal interpolation of bandlimited functions with radial basis functions, we refer the interested reader to [2], [3], [12], and [14]. A relatively complete analysis of cardinal interpolation using multiquadrics can be found in [7], Section 3 of which contains detailed proofs of Theorem I.2 and Proposition I.3.

II. Nonuniform Sampling - Univariate

Given the recovery results in the lattice case, it is interesting to consider how we may recover bandlimited functions from their samples at nonuniform point sets. We begin by considering univariate bandlimited functions. Again, the goal is to form an interpolant using radial basis functions, specifically the general multiquadrics. However, it is evident that if we take a general sequence of points \( X := (x_j)_{j \in \mathbb{Z}} \), then we cannot simply translate a single fundamental function as in the case of the integer lattice because of the irregular spacing of the points. That is, we cannot have a formula such as (1). However, we may instead consider the alternate form of the cardinal interpolant given by combining (6) and (7). Given a sequence \( X \), a radial basis function \( \phi \), and a bandlimited function \( f \in \mathbb{PW}_\pi \), we would like to form an interpolant of \( f \) from the approximation space

\[
\mathcal{A}_{\phi,X} := \left\{ \sum_{k \in \mathbb{Z}} c_k \phi(x-x_k) : (c_k) \in \mathbb{R} \right\},
\]

which satisfies \( \sum_{k \in \mathbb{Z}} c_k \phi(x_j-x_k) = f(x_j) \) for every \( j \in \mathbb{Z} \).

A. Riesz-basis Sequences

Certainly \( X \) may not be completely arbitrary or the interpolation problem above is not well-defined. It turns out that a natural condition on the sequence \( X \) is that it forms a Riesz-basis sequence for \( \mathbb{L}_2[-\pi, \pi] \) (or equivalently a complete interpolating sequence for \( \mathbb{PW}_\pi \)).

Definition II.1. A sequence \( X \subset \mathbb{R} \) is a Riesz-basis sequence for \( \mathbb{L}_2[-\pi, \pi] \) if \( (e^{-ix_j})_{j \in \mathbb{Z}} \) is a Riesz basis for \( \mathbb{L}_2[-\pi, \pi] \).

The following is a useful equivalence for Riesz-basis sequences, which can be found in [18, Theorem 9, p.143].

Theorem II.2. \( X \) is a Riesz-basis sequence for \( \mathbb{L}_2[-\pi, \pi] \) if and only if for every \( (a_j) \in \ell_2(\mathbb{Z}) \), there exists a unique \( f \in \mathbb{PW}_\pi \) such that

\[
f(x_j) = a_j, \quad j \in \mathbb{Z}.
\]

(8)

One consequence of this theorem is that if \( X \) is a Riesz-basis sequence, then for any \( f \in \mathbb{PW}_\pi \), its sample sequence \( (f(x_j))_{j \in \mathbb{Z}} \) is square-summable. The condition (8) also allows for uniqueness in the sense that if two bandlimited functions agree on the sequence \( X \), then they must be identical.

It would be nice to know if such Riesz-basis sequences are abundant or difficult to come by. First, a necessary condition for an increasing sequence \( \cdots < x_{-1} < x_0 < x_1 < \cdots \) to be
a Riesz-basis sequence for \( L_2[-\pi, \pi] \) is for it to be discrete; that is, there must exist constants \( 0 < q \leq Q < \infty \) such that \( q \leq x_{j+1} - x_j \leq Q \) for every \( j \in \mathbb{Z} \). Kadec’s 1/4-Theorem gives a sufficient condition.

**Theorem II.3 (Kadec [9]).** If

\[
\sup_{j \in \mathbb{Z}} |x_j - j| < 1/4,
\]

then \((x_j)_{j \in \mathbb{Z}}\) is a Riesz-basis sequence for \( L_2[-\pi, \pi] \).

**B. Inverse Multiquadric Interpolant**

Suppose a Riesz-basis sequence \( X \) is fixed, and that for a given bandlimited function, we would like to form a multiquadric interpolant which lies in the approximation space \( A_{\phi_{\alpha,c}}.X \). Because of the identification of \( PW_x \) with \( \ell_2(\mathbb{Z}) \) mentioned above, it suffices to show that the operator (or bi-infinite matrix) defined by \((\phi_{\alpha,c}(x_j - x_k))_{j,k \in \mathbb{Z}}\) is boundedly invertible on \( \ell_2(\mathbb{Z}) \). For a large class of so-called inverse multiquadrics, this condition is satisfied.

**Theorem II.4.** Suppose that \( X \subset \mathbb{R} \) is a Riesz-basis sequence for \( L_2[-\pi, \pi] \) and that \( \alpha < -1/2 \) and \( c > 0 \). Then the operator \( M = (\phi_{\alpha,c}(x_j - x_k))_{j,k \in \mathbb{Z}} \) is a bounded, invertible, linear operator from \( \ell_2(\mathbb{Z}) \) to \( \ell_2(\mathbb{Z}) \). Consequently, given \( f \in PW_x \), there is a unique interpolant

\[
I_{\alpha,c}.f(x) = \sum_{k \in \mathbb{Z}} a_k \phi_{\alpha,c}(x - x_k) \in A_{\phi_{\alpha,c}}.X
\]

such that \( I_{\alpha,c}.f(x_j) = f(x_j) \) for every integer \( j \). Moreover, \( a_k = (M^{-1}y)_k \), where \( y_k = f(x_k) \).

The reason for the condition that \( \alpha < -1/2 \) is two-fold. It implies that \( \phi_{\alpha,c} \) is integrable and so the Fourier inversion theorem holds, which one can use to show injectivity of the multiquadric matrix \( M \) via the Riesz basis condition, and it also allows one to prove boundedness of \( M \) by appealing to Schur’s Test.

**C. Recovery Results**

As before, we are interested in recovering bandlimited functions \( f \) from their samples using the multiquadric interpolants. Similar to the lattice case, we have the following.

**Theorem II.5.** Let \( \alpha < -1/2 \) and let \( X \) be a Riesz-basis sequence for \( L_2[-\pi, \pi] \). If \( f \in PW_x \), then \( I_{\alpha,c}.f \in L_2(\mathbb{R}) \),

\[
\lim_{c \to \infty} \| I_{\alpha,c}.f - f \|_{L_2(\mathbb{R})} = 0,
\]

and

\[
\lim_{c \to \infty} | I_{\alpha,c}.f(x) - f(x) | = 0
\]

uniformly on \( \mathbb{R} \).

There are a few considerations necessary to prove Theorem II.5. One must show that the multiquadric interpolation operator \( I_{\alpha,c} \cdot PW_x \to L_2(\mathbb{R}) \) is uniformly bounded in \( c \). Consider the Fourier transform of \( I_{\alpha,c}.f \), which is given by

\[
\tilde{I}_{\alpha,c}.f(\xi) = \hat{\phi}_{\alpha,c}(\xi) \sum_{k \in \mathbb{Z}} a_k e^{-i\xi x_k} =: \hat{\phi}_{\alpha,c}(\xi) \Psi_c(\xi).
\]

To estimate the \( L_2 \) norm of the interpolant, one needs to relate the norm of the nonharmonic Fourier series, \( \Psi_c \), to the norm of \( f \). To do this, one takes considerable advantage of the fact that the function \( \Psi_c \) is globally defined by its Riesz basis representation on the interval \([-\pi, \pi]\). As a consequence, in formulating the periodization argument used to estimate the norm of \( I_{\alpha,c}.f \), one can relate its norm on the rest of the line to its norm on \([-\pi, \pi]\). The proof of Theorem II.5 follows the same techniques as [15], and can also be synthesized from the results in [5] and [11].

**D. Fundamental Functions**

Due to the nonuniform nature of the point sets we have been considering, the multiquadric interpolants are most readily seen by inverting the multiquadric matrix, and thus the samples \((f(x_k))_{k \in \mathbb{Z}}\) are implicit in the formula for \( I_{\alpha,c}.f \). However, similar to the cardinal interpolation setting, we may rewrite the interpolant in such a way as to have the samples appear explicitly. Precisely, for each \( k \in \mathbb{Z} \), we form a fundamental function \( L_k \) which satisfies \( L_k(x_j) = \delta_{k,j} \), \( j \in \mathbb{Z} \). In general, each fundamental function will be different since the points are not symmetric. The fundamental functions are of the form

\[
L_k(x) = \sum_{k \in \mathbb{Z}} d_k \phi_{\alpha,c}(x - x_k), \quad x \in \mathbb{R},
\]

where the coefficients \( d_k \) come from applying \( M^{-1} \) to the canonical unit vector element \( e_k \) of \( \ell_2(\mathbb{Z}) \). Thus, given \( f \in PW_x \), we have the following alternate representation of its interpolant:

\[
I_{\alpha,c}.f(x) = \sum_{k \in \mathbb{Z}} f(x_k) L_k(x), \quad x \in \mathbb{R}.
\]

For further results along the lines of those in this section, the reader is invited to consult [12] and [15]. Another example of a radial basis function \( \phi \) that can be used to form interpolants in the space \( A_{\phi,X} \) is the Gaussian kernel \( e^{-\lambda|x|^2} \) for some \( \lambda > 0 \). In that case, the analogous result to Theorem II.5 holds when \( \lambda \to 0^+ \), [15, Theorems 4.3 and 4.4].

**III. NONUNIFORM SAMPLING - MULTIVARIATE**

As mentioned in Section II-A, Riesz bases of exponentials for \( L_2[-\pi, \pi] \) are fairly easy to come by. In contrast, finding such bases for \( L_2 \) spaces in higher dimensions is much more difficult, and depends heavily on the geometry of the underlying set. Suppose that \( S \subset \mathbb{R}^d \) has positive Lebesgue measure. Then as before, define the Paley-Wiener space \( PW_S \) to be those \( L_2(\mathbb{R}^d) \) functions whose Fourier transforms are supported on \( S \). The multivariate version of Theorem II.2 is true, and so to consider interpolation in higher dimensions, we need to find Riesz bases.

For Paley-Wiener spaces over cubes in \( d \) dimensions, there are some extensions of Kadec’s 1/4-Theorem, and so at least small perturbations of the multi-integer lattice give Riesz-basis sequences for \( L_2[-\pi, \pi]^d \). Consequently, one might try to extend the proof of Theorem II.5 to such spaces in higher dimensions. However, the correct geometry for the proof is to
consider Paley-Wiener spaces over balls in higher dimensions; it just happens that in one dimension these shapes are one and the same. It should not be a surprise, however, that Euclidean balls are the correct geometry given that our interpolation scheme uses radial functions.

The natural version of Theorem II.5 holds in the case where $X$ is a Riesz-basis sequence for $L_2(B_2)$ where $B_2$ is the Euclidean ball in $\mathbb{R}^d$. However, this theorem may well be vacuous considering it is still unknown whether $L_2(B_2)$ admits a Riesz basis of exponentials even in two dimensions. Much work has been done on the problem of finding Riesz bases of exponentials for different sets, including developments by Lyubarskii and Rashkovskii [13], Grepstad and Lev [4], and simultaneously Kolountzakis [10]. One of the results in these papers is that for zonotopes, which are convex bodies in $\mathbb{R}^d$ whose faces are symmetric about the origin, one can find Riesz bases of exponentials. Moreover, given any $\delta < 1$, one can find a zonotope $Z$ such that $\delta B_2 \subset Z \subset B_2$. Therefore, one way to find a recovery result that we know to be meaningful is to approximate the Euclidean ball by such zonotopes, and consider bandlimited functions whose support lies in a smaller ball $\beta B_2 \subset \delta B_2$.

A. Multivariate Recovery

Suppose that $0 < \beta < \delta < 1$, and $Z$ is some convex body such that $\beta B_2 \subset \delta B_2 \subset Z \subset B_2$. Assume also that $L_2(Z)$ has a Riesz basis of exponentials $(e^{-i\langle x_k, \cdot \rangle})_{k \in \mathbb{Z}}$ (per the discussion in the previous paragraph, this depends on the geometry of $Z$). As in the univariate case, we aim to interpolate bandlimited functions at the sequence $(x_k)$. We will exploit the geometry of the problem to form an interpolant for functions $f \in PW_{\beta B_2}$ of the form

$$I_{\alpha,c}f(x) = \sum_{k \in \mathbb{Z}^d} a_k \phi_{\alpha,c}(x - x_k), \quad x \in \mathbb{R}^d.$$  

The interpolant is formed, as in the univariate case, by inverting the multiquadric matrix. Moreover, the desired convergence results hold, again for a large family of inverse multiquadratics.

Theorem III.1. Let $\alpha < -d/2$. Suppose that $\delta \in (2/3, 1)$ and $\beta \in (0, 3\delta - 2)$. Suppose that $Z$ is a symmetric convex body such that $\delta B_2 \subset Z \subset B_2$ and that $(e^{-i\langle x_k, \cdot \rangle})_{k \in \mathbb{Z}}$ is a Riesz basis for $L_2(Z)$. Then for every $f \in PW_{\beta B_2}$,

$$\lim_{c \to \infty} \|I_{\alpha,c}f - f\|_{L_2(\mathbb{R}^d)} = 0,$$

and

$$\lim_{c \to \infty} |I_{\alpha,c}f(x) - f(x)| = 0$$

uniformly on $\mathbb{R}^d$.

The restriction on $\delta$ is needed because $\beta$ must be positive, while the range for $\beta$ comes from estimating maximum or minimum values of the multiquadric on the different bodies. Theorem III.1 is a special case of the main result in [6]. Similar theorems hold when using the Gaussian [1], or the more general class of functions $g_\alpha(x) = \int_{\mathbb{R}^d} e^{-\alpha \|\xi\|^2} e^{i\langle \xi, x \rangle} d\xi$.

IV. APPROXIMATION RATES

Upon finding a convergence result, it is natural to ask how quickly convergence is achieved. In many of the situations we have explored thus far, there are answers to this question. Since most of our results are on convergence as the multiquadric shape parameter $\sigma$ tends to infinity, we would like to find rates in terms of this parameter.

Our first result is an oversampling one in the univariate case.

Theorem IV.1. Suppose that $f \in PW_\sigma$ for some $\sigma < \pi$, and that $X$ is a Riesz-basis sequence for $L_2[-\pi, \pi]$. If $\alpha < -1/2$, then

$$\|I_{\alpha,c}f - f\|_{L_2(\mathbb{R})} \leq C e^{-c(\pi - \sigma)} \|f\|_{L_2(\mathbb{R})},$$

where the constant $C$ depends on $\alpha$ and $X$, but not on $c$.

We achieve similar rates in the multivariate case, again since we are recovering functions with smaller band than the set we have a Riesz-basis sequence for. The proof of the following theorem may be found in [6].

Theorem IV.2. Suppose that $\alpha, \delta, \beta, Z$, and $X$ are as in Theorem III.1. Then there exists a constant $C > 0$ such that for every $f \in PW_{\beta B_2}$,

$$\|I_{\alpha,c}f - f\|_{L_2(\mathbb{R}^d)} \leq C \left(\frac{\beta}{\delta}\right)^{\alpha + \frac{d+1}{2}} e^{-c(\delta - 2 - \beta)} \|f\|_{L_2(\mathbb{R}^d)},$$

where $C$ is independent of $c$.

We note that the conditions on $\beta$ and $\delta$ in Theorem III.1 imply that the exponent on the right hand side of (13) is indeed negative. In both cases, the fact that we are oversampling yields exponential convergence rates in terms of the shape parameter $c$, which is similar to what happens when we oversample in the classical sampling formula. However, it is difficult to assess the convergence rates for $I_{\alpha,c}f$ in the case that $f$ has full band (i.e. when we are not oversampling). Nonetheless, there are some results along a different line of reasoning.

Let us now turn our gaze to fixing a sequence $X$ which is a Riesz-basis sequence for $L_2[-\pi, \pi]$, as well as fixing a parameter $h \in (0, 1]$. The question we consider is what happens when we form an inverse multiquadric interpolant which matches a given function $g$ at the tighter sequence $hX$? That is, we wish to find how close our interpolant is to the given function in terms of the parameter $h$. This is related to much of what is done in traditional radial basis function theory and numerical analysis, where $h$ may be thought of as a mesh parameter. It was shown in [5] that one can actually interpolate Sobolev functions $g \in W^k_p(\mathbb{R})$, that is the space of $L_2(\mathbb{R})$ functions whose first $k$ weak derivatives are also square integrable. The inverse multiquadric interpolant takes a slightly different form, namely

$$I^hXg(x) = \sum_{k \in \mathbb{Z}} a_k \phi_{\alpha,c}^h \left(\frac{x}{h} - x_k\right).$$

(14)
To achieve the proper form of the interpolant, the multi-quadric shape parameter is related to the mesh parameter by the relation \( c = 1/h \).

**Theorem IV.3.** Suppose that \( \alpha < -1/2 \), \( k \in \mathbb{N} \), \( 0 < h \leq 1 \), and \( X \) is a Riesz-basis sequence for \( L_2[-\pi, \pi] \). Then there exists a constant \( C \) independent of \( h \) such that for every \( g \in W_2^k(\mathbb{R}) \),

\[
\| I^{h;X} g - g \|_{L_2(\mathbb{R})} \leq C h^k \| g \|_{W_2^k(\mathbb{R})}.
\]  

(15)

Of course, this theorem holds in the case where \( X \) is the integer lattice. The lattice case merits special attention and recent joint work with Ledford has shown that similar approximation rates hold for interpolation at \( h \mathbb{Z} \) even with a broader range of \( \alpha \). In that case, one can apply Fourier multiplier theory to attain \( L_p \) rates for functions \( g \in W_2^p(\mathbb{R}) \). Multivariate extensions should be similar following the techniques of [8], with possibly more restrictions on \( \alpha \).

Theorem IV.3 and other approximation rates along similar lines (e.g. for Gaussian interpolation) can be found in [5].

V. NUMERICAL EXAMPLES

To give some brief numerical results, we focus on univariate cardinal interpolation as in Section I. First note that Proposition I.3 implies that \( I_{\alpha,c} \) must converge to the sinc function. Figure 1 shows the graph of \( L_{-1,c} \) (the fundamental function associated with the Poisson kernel) for different values of \( c \). As expected, for the larger value, \( c = 10 \), the accuracy is much higher. The maximum of the difference of \( L_{-1,c} \) and the sinc function on the interval \([-10, 10]\) considered in the figure was .1174 when \( c = 1 \) and .0098 when \( c = 10 \).

![Fig. 1. Plots of sinc function and Fundamental function for the Poisson kernel (with \( \alpha = -1 \)) with shape parameters \( c = 1 \) (left) and \( c = 10 \) (right).](image1)

To calculate \( I_{\alpha,c} g \), the series in (7) was truncated at \( k = \pm10 \). It would appear from Table I that for low values of \( c \), it is beneficial to take a large positive value of \( \alpha \). However, for large values of \( c \), this advantage appears to be lost, likely either because \( c \) is very large compared to \( |\alpha| \), or due to truncation error in approximating the interpolant.

As mentioned before, the use of the multiquadrics here is a special case of a more general phenomenon. Most of the results contained here can be obtained with other radial basis functions including the Gaussian, and even other more general functions.

### REFERENCES


[2] B. J. C. Baxter, The asymptotic cardinal function of the multiquadrics here is a special case of a more general phenomenon. Most of the results contained here can be obtained with other radial basis functions including the Gaussian, and even other more general functions.

### TABLE I

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<td>3645</td>
<td>.0313</td>
<td>.00154</td>
</tr>
</tbody>
</table>

To calculate \( I_{\alpha,c} g \), the series in (7) was truncated at \( k = \pm10 \). It would appear from Table I that for low values of \( c \), it is beneficial to take a large positive value of \( \alpha \). However, for large values of \( c \), this advantage appears to be lost, likely either because \( c \) is very large compared to \( |\alpha| \), or due to truncation error in approximating the interpolant.

As mentioned before, the use of the multiquadrics here is a special case of a more general phenomenon. Most of the results contained here can be obtained with other radial basis functions including the Gaussian, and even other more general functions.

### REFERENCES


[2] B. J. C. Baxter, The asymptotic cardinal function of the multiquadrics here is a special case of a more general phenomenon. Most of the results contained here can be obtained with other radial basis functions including the Gaussian, and even other more general functions.

### TABLE I

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( c = 1 )</th>
<th>( c = 10 )</th>
<th>( c = 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 7/2 )</td>
<td>1047</td>
<td>.0223</td>
<td>.00156</td>
</tr>
<tr>
<td>( \alpha = 1/2 )</td>
<td>1818</td>
<td>.0599</td>
<td>.00135</td>
</tr>
<tr>
<td>( \alpha = -1/2 )</td>
<td>2227</td>
<td>.0727</td>
<td>.00135</td>
</tr>
<tr>
<td>( \alpha = -1 )</td>
<td>2452</td>
<td>.0728</td>
<td>.00135</td>
</tr>
<tr>
<td>( \alpha = -7/2 )</td>
<td>3645</td>
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<td>.00154</td>
</tr>
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