

The dual frame induced by an invertible frame multiplier

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Abstract—The inverse of an invertible frame multiplier can be represented as a multiplier with reciprocal symbol and sequences dual to the original frames. One of the duals is unique with the property that the other one can be arbitrarily chosen, which is shown in this paper for several cases. We also show when the dual induced by the invertible frame multiplier is the canonical dual and when it has the same structure as the original dual.

I. INTRODUCTION

Frame multipliers [1], i.e. operators that consists of analysis, multiplication by a fixed symbol and synthesis, play an important role in a lot of scientific fields. For example in acoustics they are used as a way to implement time-varying filters [2], [3], and were recently applied to applications in psychoacoustics [4] and denoising [5]. In recent years they also became a prominent topic in mathematical analysis [6], [7], [8].

In particular the investigation on invertibility of multipliers lead to several papers [9], [10], [11], [12], [13], some of which leading to interesting results on dual frames. Here we concentrate on the representation of an invertible frame multiplier. Any such operator can be again represented as a multiplier, using dual frames of the original frames and the reciprocal symbol. Either one of those dual frames is unique with the property that the other one can be arbitrarily chosen. In this contribution we focus on the properties of these unique sequences, which we call the *dual frames induced by the invertible frame multiplier*.

In the next section we collect basic definitions and notations. In Section III we state the above mentioned representation result for semi-normalized symbols. In Section IV we regard cases, where the symbol is not necessarily semi-normalized anymore. And finally in Section V we look at the structure of the dual frames induced by an invertible frame multiplier: When do they keep a given structure of the original frame, when do they coincide with the canonical dual frame?

II. BASIC DEFINITIONS AND NOTATION

The letter \mathcal{H} denotes a separable Hilbert space; Φ and Ψ denote the sequences $(\phi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$, respectively, with elements from \mathcal{H} ; m denotes a complex scalar sequence $(m_n)_{n=1}^{\infty}$ and \bar{m} denotes the sequence $(\bar{m}_n)_{n=1}^{\infty}$ consisting of the complex conjugates of the elements of m ; $1/m$ denotes the

sequence $(1/m_n)_{n=1}^{\infty}$; $m\Phi$ denotes the sequence $(m_n\phi_n)_{n=1}^{\infty}$. The scalar sequence m is called *semi-normalized* if $0 < \inf_n |m_n| \leq \sup_n |m_n| < \infty$. When the index set is omitted, the set \mathbb{N} should be understood. An operator $L : \mathcal{H} \rightarrow \mathcal{H}$ is called *invertible* if it is a bounded bijection from \mathcal{H} onto \mathcal{H} .

Recall that Φ is called:

- a *Bessel sequence* in \mathcal{H} , if there exists a positive constant B so that $\sum_n |\langle h, \phi_n \rangle|^2 \leq B \|h\|^2$ for all $h \in \mathcal{H}$;
- a *frame* for \mathcal{H} if there exist positive constants A and B so that $A \|h\|^2 \leq \sum_n |\langle h, \phi_n \rangle|^2 \leq B \|h\|^2$ for all $h \in \mathcal{H}$;
- a *Riesz basis* for \mathcal{H} if it is a frame for \mathcal{H} and a Schauder basis of \mathcal{H} at the same time.

For a Bessel sequence Φ in \mathcal{H} , U_{Φ} will denote the analysis operator (i.e., the operator given by $U_{\Phi}f = (\langle f, \phi_n \rangle)$, $f \in \mathcal{H}$) and $\mathcal{R}(U_{\Phi})$ will denote the range of U_{Φ} .

Two frames Φ and Ψ for \mathcal{H} are called *equivalent* if there is an invertible operator L from \mathcal{H} onto \mathcal{H} so that $L\phi_n = \psi_n$ for every $n \in \mathbb{N}$. When Φ is a frame for \mathcal{H} , then a frame $(g_n)_{n=1}^{\infty}$ for \mathcal{H} satisfying $f = \sum_n \langle f, \phi_n \rangle g_n = \sum_n \langle f, g_n \rangle \phi_n$ for every $f \in \mathcal{H}$ is called a *dual frame of Φ* ; the operator S , given by $Sf := \sum_n \langle f, \phi_n \rangle \phi_n$, is well defined on \mathcal{H} and invertible, and the sequence $(S^{-1}\phi_n)$ is a dual frame of Φ called the *canonical dual of Φ* . For more on frames we refer to [14], [15], [16], [17].

In addition to dual frames, we will also use sequences, which are not necessarily frames and satisfy a reconstruction formula. Given a frame Φ for \mathcal{H} , the sequence $F = (f_n)$ with elements from \mathcal{H} is called:

- an *analysis pseudo-dual* (in short, *a-pseudo-dual*) of Φ , if $\sum \langle f, f_n \rangle \phi_n = f$ for every $f \in \mathcal{H}$;
- a *synthesis pseudo-dual* (in short, *s-pseudo-dual*) of Φ , if $\sum_n \langle f, \phi_n \rangle f_n = f$ for every $f \in \mathcal{H}$.

For more on a- and s-pseudo duals of frames, we refer to [18].

Given Φ, Ψ , and m , the operator $M_{m, \Phi, \Psi}$ given by

$$M_{m, \Phi, \Psi} f = \sum_n m_n \langle f, \psi_n \rangle \phi_n, \quad f \in \text{dom}(M_{m, \Phi, \Psi}),$$

is called a *multiplier* and m is called the *symbol* of the multiplier. When Φ and Ψ are frames (resp. Riesz bases), $M_{m, \Phi, \Psi}$ is called a *frame multiplier* (resp. *Riesz multiplier*). A multiplier $M_{m, \Phi, \Psi}$ with m being the constant sequence (c, c, c, \dots) will be denoted by $M_{(c), \Phi, \Psi}$.

III. INVERTIBLE FRAME MULTIPLIERS WITH SEMI-NORMALIZED SYMBOL

In this section we consider frame multipliers with semi-normalized symbols.

A. The case of Riesz multipliers

When both of the frames of the multiplier are Riesz bases, then the multiplier is invertible and the inverse has a nice representation using the canonical duals:

Proposition III.1: [1] If m is semi-normalized, then a Riesz multiplier $M_{m,\Phi,\Psi}$ is invertible and

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\tilde{\Psi},\tilde{\Phi}}, \quad (1)$$

where $\tilde{\Phi}$ and $\tilde{\Psi}$ are the canonical duals of Φ and Ψ , respectively.

B. The case of frame multipliers

The result on Riesz multipliers (Proposition III.1) naturally leads to the question

[Q1] *Are there other invertible frame multipliers $M_{m,\Phi,\Psi}$ whose inverses can be written as in (1)?*

and to the less restrictive question

[Q2] *Are there other invertible frame multipliers $M_{m,\Phi,\Psi}$ whose inverses can be represented using the inverted symbol $1/m$ and appropriate dual frames of Φ and Ψ (not necessarily the canonical duals)?*

The answer to Q2 is affirmative for any invertible frame multiplier with semi-normalized symbol (for examples of such multipliers, see the paragraph after Proposition III.3):

Theorem III.2: [13] Let Φ and Ψ be frames for \mathcal{H} , and let the symbol m be semi-normalized. Assume that $M_{m,\Phi,\Psi}$ is invertible. Then the following statements hold.

- There exists a unique dual frame Φ^\dagger of Φ , so that

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^d,\Phi^\dagger}, \quad \forall \text{ dual frames } \Psi^d \text{ of } \Psi. \quad (2)$$

- There exists a unique dual frame Ψ^\dagger of Ψ , so that

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^\dagger,\Phi^d}, \quad \forall \text{ dual frames } \Phi^d \text{ of } \Phi. \quad (3)$$

- Ψ^\dagger is the only Bessel sequence in \mathcal{H} which satisfies (3).
- Φ^\dagger is the only Bessel sequence in \mathcal{H} which satisfies (2).
- If $G = (g_n)$ is a Bessel sequence in \mathcal{H} which fulfills

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^\dagger,G},$$

then G must be a dual frame of Φ .

- If $F = (f_n)$ is a Bessel sequence in \mathcal{H} which fulfills

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,F,\Phi^\dagger},$$

then F must be a dual frame of Ψ .

Theorem III.2 naturally leads to the following question:

[Q3] *Are there non-Bessel sequences, which can be used in the role of Φ^d and Ψ^d , or in the role of Φ^\dagger and Ψ^\dagger ?*

We answer the question in the following statement:

Proposition III.3: Let Φ and Ψ be frames for \mathcal{H} , and let the symbol m satisfy $0 < \inf_n |m_n| \leq \sup_n |m_n| < \infty$. Assume that $M_{m,\Phi,\Psi}$ is invertible and let Φ^\dagger and Ψ^\dagger be from Theorem III.2. Then the following statements hold.

- Ψ^\dagger is the only sequence in \mathcal{H} which satisfies (3).
- Φ^\dagger is the only sequence in \mathcal{H} which satisfies (2).
- $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^{sd},\Phi^\dagger}$, \forall s-pseudo-duals Ψ^{sd} of Ψ .
- $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^\dagger,\Phi^{ad}}$, \forall a-pseudo-duals Φ^{ad} of Φ .
- If $G = (g_n)$ is a sequence in \mathcal{H} such that $M_{1/m,\Psi^\dagger,G}$ is well-defined and

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^\dagger,G},$$

then G must be an a-pseudo-dual of Φ .

- If $F = (f_n)$ is a sequence in \mathcal{H} such that $M_{1/m,F,\Phi^\dagger}$ is well-defined and

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,F,\Phi^\dagger},$$

then F must be an s-pseudo-dual of Ψ .

Question Q1 also has an affirmative answer. For example, every frame multiplier $M_{(1),\Phi,\Phi}$ (being the frame operator for the frame Φ) is invertible and its inverse satisfies (1). Unlike the case with Q2 and as expected, not every invertible frame multiplier with semi-normalized symbol satisfies (1). As an example, consider the frame multiplier $M_{(1),\Phi,\Psi}$ with $\Phi = (e_1, e_1, e_1, e_2, e_2, e_2, e_3, e_3, e_3, \dots)$ and $\Psi = (e_1, e_1, -e_1, e_2, e_2, -e_2, e_3, e_3, -e_3, \dots)$, [13]. The following statement gives a class of multipliers, which are invertible and the inverse satisfies the formula (1). Notice that while Theorem III.2 contains the invertibility of the multiplier as an assumption, the next proposition gives a sufficient condition for the invertibility, resp. non-invertibility, of the multiplier.

Proposition III.4: [13, Prop. 4.3] Let Φ and Ψ be frames for \mathcal{H} and $(m_n) = (c)$, $c \neq 0$. Then the following assertions hold.

- If $\mathcal{R}(U_\Phi) \subseteq \mathcal{R}(U_\Psi)$, then $M_{(1/c),\tilde{\Psi},\tilde{\Phi}}$ is a bounded right inverse of $M_{(c),\Phi,\Psi}$.
- If $\mathcal{R}(U_\Psi) \subseteq \mathcal{R}(U_\Phi)$, then $M_{(1/c),\tilde{\Psi},\tilde{\Phi}}$ is a bounded left inverse of $M_{(c),\Phi,\Psi}$.
- If $\mathcal{R}(U_\Phi) = \mathcal{R}(U_\Psi)$, then $M_{(c),\Phi,\Psi}$ is invertible and $M_{(c),\Phi,\Psi}^{-1} = M_{(\frac{1}{c}),\tilde{\Psi},\tilde{\Phi}}$.
- If $\mathcal{R}(U_\Phi) \subsetneq \mathcal{R}(U_\Psi)$, then $M_{(c),\Phi,\Psi}$ is not invertible.
- If $\mathcal{R}(U_\Psi) \subsetneq \mathcal{R}(U_\Phi)$, then $M_{(c),\Phi,\Psi}$ is not invertible.

Let Φ and Ψ be frames for \mathcal{H} . Note that the condition $\mathcal{R}(U_\Phi) = \mathcal{R}(U_\Psi)$ corresponds to Φ and Ψ being equivalent frames [14, Corollary 4.5], while the condition $\mathcal{R}(U_\Phi) \subseteq \mathcal{R}(U_\Psi)$ corresponds to Ψ being partially equivalent to Φ , see [19].

As Proposition III.4(iii) determines $\mathcal{R}(U_\Phi) = \mathcal{R}(U_\Psi)$ as a sufficient condition for invertibility and validity of formula (1), it is natural to ask whether this condition is necessary and the answer is yes:

Theorem III.5: [13] Let Φ and Ψ be frames for \mathcal{H} and $(m_n) = (c)$, $c \neq 0$. The following statements are equivalent.

- (a) $M_{(c),\Phi,\Psi}$ is invertible and $M_{(c),\Phi,\Psi}^{-1} = M_{(\frac{1}{c}),\tilde{\Psi},\tilde{\Phi}}$.
(b) $\mathcal{R}(U_\Phi) = \mathcal{R}(U_\Psi)$.

IV. INVERTIBLE FRAME MULTIPLIERS WITH *not* NECESSARILY SEMI-NORMALIZED SYMBOLS

The results in the previous section concern the case when the symbol of the multiplier is semi-normalized. One naturally may ask what happens if the symbol is not semi-normalized. Here we present results in this direction.

A. The Riesz multiplier case

If the symbol of a Riesz multiplier is *not* semi-normalized, the multiplier can never be invertible:

Proposition IV.1: [9, Theor. 5.1] Let Φ and Ψ be Riesz bases for \mathcal{H} . Then the multiplier $M_{m,\Phi,\Psi}$ is invertible if and only if m is semi-normalized.

B. The frame multiplier case

For frames the above mentioned result is not true, the redundancy allows inversion also for not semi-normalized symbols. See the examples below. We can state:

Proposition IV.2: Let Φ and Ψ be frames for \mathcal{H} , and let the symbol m be such that $m_n \neq 0$ for every n and the sequence $m\Phi$ is a frame for \mathcal{H} . Assume that $M_{m,\Phi,\Psi}$ is invertible. Then the following statements hold.

- There exists a unique sequence Ψ^\dagger in \mathcal{H} so that

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^\dagger,\Phi^{ad}}, \quad \forall \text{ a-pseudo-duals } \Phi^{ad} \text{ of } \Phi,$$

and it is a dual frame of Ψ .

- If $G = (g_n)$ is a sequence in \mathcal{H} such that $M_{1/m,\Psi^\dagger,G}$ is well-defined and

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^\dagger,G},$$

then G must be an a-pseudo-dual of Φ .

The above proposition is related to properties of the unique dual frame Ψ^\dagger of Ψ . In a similar way, one can relate such a result to Φ and its unique dual frame induced by an invertible frame multiplier:

Proposition IV.3: Let Φ and Ψ be frames for \mathcal{H} , and let the symbol m be such that $m_n \neq 0$ for every n and the sequence $m\Psi$ is a frame for \mathcal{H} . Assume that $M_{m,\Phi,\Psi}$ is invertible. Then the following statements hold.

- There exists a unique sequence Φ^\dagger in \mathcal{H} so that

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^{sd},\Phi^\dagger}, \quad \forall \text{ s-pseudo-duals } \Psi^{sd} \text{ of } \Psi,$$

and it is a dual frame of Φ .

- If $F = (f_n)$ is a sequence in \mathcal{H} such that $M_{1/m,F,\Phi^\dagger}$ is well-defined and

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,F,\Phi^\dagger},$$

then F must be an s-pseudo-dual of Ψ .

A slightly different, not equivalent statement is the following:

Proposition IV.4: Let Φ and Ψ be frames for \mathcal{H} , and let the symbol m be such that $m_n \neq 0$ for every n and $m \in \ell^\infty$.

Assume that $M_{m,\Phi,\Psi}$ is invertible. Then the conclusions of Propositions IV.2 and IV.3 hold.

Consider the following examples for a comparison of the above results:

Example IV.5: Consider

$$\begin{aligned} \Phi &= (e_1, e_2, \frac{1}{2}e_1, e_3, \frac{1}{\sqrt{2}}e_1, e_4, \frac{1}{\sqrt{3}}e_1, e_5, \dots), \\ \Psi &= (e_1, e_2, \frac{1}{\sqrt{2}}e_1, e_3, \frac{1}{\sqrt{2^2}}e_1, e_4, \frac{1}{\sqrt{2^3}}e_1, e_5, \dots), \\ m &= (1, 1, \sqrt{2}, 1, \sqrt{2^2}, 1, \sqrt{2^3}, 1, \dots). \end{aligned}$$

Then $M_{m,\Phi,\Psi} = Id_{\mathcal{H}}$ and all the assumptions of Proposition IV.2 are satisfied. Since $m \notin \ell^\infty$, Proposition IV.4 does not apply.

Example of a multiplier, for which both Proposition IV.4 and Proposition IV.2 can be applied.

Example IV.6: [11] Consider

$$\begin{aligned} \Phi &= (e_1, e_1, -e_1, e_2, e_2, -e_2, e_3, e_3, -e_3, \dots), \\ \Psi &= (e_1, e_1, e_1, e_2, \frac{1}{2}e_2, \frac{1}{2}e_2, e_3, \frac{1}{3}e_3, \frac{1}{3}e_3, \dots), \\ m &= (1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{3}, \frac{1}{3}, \dots). \end{aligned}$$

Then $M_{m,\Phi,\Psi} = Id_{\mathcal{H}}$ and all the assumptions of Propositions IV.2 and IV.4 are satisfied.

Note that there are no cases, where Proposition IV.4 can be applied and Propositions IV.2 and IV.3 can not, by [9, Prop. 3.1(iv)].

V. THE STRUCTURE OF THE DUAL FRAME INDUCED BY AN INVERTIBLE FRAME MULTIPLIER

In this section we give some results related to the following questions:

[Q4] When the dual frame Φ^\dagger (resp. Ψ^\dagger) of Φ (resp. Ψ) from the statements in Section III is exactly the canonical dual?

[Q5] If Φ (resp. Ψ) is a frame with specific structure, does the dual frame Φ^\dagger (resp. Ψ^\dagger) have the same kind of structure?

A. Concerning Q4

Proposition V.1: [13] Under the assumptions of Theorem III.2, Ψ^\dagger is the canonical dual of Ψ if and only if Ψ is an equivalent frame to $(m_n\phi_n)$ and Φ^\dagger is the canonical dual of Φ if and only if Φ is an equivalent frame to $(\overline{m_n}\psi_n)$.

Concerning a representation of $M_{(1),\Phi,\Psi}^{-1}$ using both canonical dual frames of Φ and Ψ , the following result is valid.

Proposition V.2: [13] Let the assumptions of Theorem III.2 hold and let $\tilde{\Phi}$ and $\tilde{\Psi}$ denote the canonical duals of Φ and Ψ , respectively. Then

$$M_{m,\Phi,\Psi}^{-1} = M_{m,\tilde{\Psi},\tilde{\Phi}} \Leftrightarrow \Psi \text{ is equivalent to } m\Phi \Leftrightarrow \Psi^\dagger = \tilde{\Psi};$$

$$M_{m,\Phi,\Psi}^{-1} = M_{m,\tilde{\Psi},\tilde{\Phi}} \Leftrightarrow \Phi \text{ is equivalent to } \overline{m}\Psi \Leftrightarrow \Phi^\dagger = \tilde{\Phi}.$$

For the case $m = (1)$,

$$\begin{aligned} M_{(1),\Phi,\Psi}^{-1} = M_{(1),\tilde{\Psi},\tilde{\Phi}} \Leftrightarrow \Psi \text{ is equivalent to } \Phi &\Leftrightarrow \Psi^\dagger = \tilde{\Psi} \\ &\Leftrightarrow \Phi^\dagger = \tilde{\Phi}. \end{aligned}$$

B. Concerning Q5

In this section we consider a class of frame multipliers, for which we obtain affirmative answer to question Q5. We define a class of frames as follows:

Definition V.3: [13] Let Λ be a discrete set. Consider $\theta : \Lambda \rightarrow \mathcal{B}(\mathcal{H})$ (i.e., $\theta(\lambda)$ being a bounded operator from \mathcal{H} into \mathcal{H} for $\lambda \in \Lambda$). Assume that there exist $\phi : \Lambda \times \Lambda \rightarrow \mathbb{C}$ and $\mu : \Lambda \times \Lambda \rightarrow \Lambda$ such that for every $\lambda \in \Lambda$,

the mapping $\lambda' \rightarrow \mu(\lambda, \lambda')$ is a bijection from Λ onto Λ ,

$$\begin{aligned}\theta(\lambda)^* \theta(\lambda') &= \phi(\lambda, \lambda') \theta(\mu(\lambda, \lambda')), \quad \forall \lambda' \in \Lambda, \\ \theta(\lambda) \theta(\mu(\lambda, \lambda')) &= \overline{\phi(\lambda, \lambda')} \theta(\lambda'), \quad \forall \lambda' \in \Lambda.\end{aligned}$$

Then a sequence of the form $(g_\lambda)_{\lambda \in \Lambda} = (\theta(\lambda)g)_{\lambda \in \Lambda}$, $g \in \mathcal{H}$, $g \neq 0$, is called a θ -pseudo-coherent system.

A θ -pseudo-coherent system which is a frame for \mathcal{H} is called a θ -pseudo-coherent frame for \mathcal{H} . A multiplier for θ -pseudo-coherent frames is called a θ -pseudo-coherent frame multiplier. When $m = (1)$, a θ -pseudo-coherent frame multiplier is also called θ -pseudo-coherent frame-type operator or θ -pseudo-coherent mixed frame operator.

Remark V.4: The class of Θ -pseudo-coherent frames includes Gabor frames and coherent frames. For details, see [11].

First we note that the canonical dual of a θ -pseudo-coherent frame is also a θ -pseudo-coherent frame:

Proposition V.5: [13] Let $g \in \mathcal{H}$ and $(g_\lambda)_{\lambda \in \Lambda} = (\theta(\lambda)g)_{\lambda \in \Lambda}$ be a θ -pseudo-coherent frame for \mathcal{H} . Then the frame operator S_g for $(g_\lambda)_{\lambda \in \Lambda}$ commutes with all the operators $\theta(\lambda)$, $\lambda \in \Lambda$, and thus the canonical dual of $(g_\lambda)_{\lambda \in \Lambda}$ is the θ -pseudo-coherent frame $(\theta(\lambda)\tilde{g})_{\lambda \in \Lambda}$ with $\tilde{g} = S_g^{-1}g$.

For an invertible θ -pseudo-coherent frame-type operator, we give an affirmative answer to question Q5:

Proposition V.6: [13] Let the θ -pseudo-coherent frame-type operator $V = M_{(1),(\theta(\lambda)v)_{\lambda \in \Lambda},(\theta(\lambda)u)_{\lambda \in \Lambda}}$ be invertible. Then the frames $(\theta(\lambda)v)_{\lambda \in \Lambda}^\dagger$ and $(\theta(\lambda)u)_{\lambda \in \Lambda}^\dagger$ from Theorem III.2 have a θ -pseudo-coherent structure. Furthermore, V^{-1} can be written as a θ -pseudo-coherent frame-type operator as follows:

$$\begin{aligned}V^{-1} &= M_{(1),(\theta(\lambda)V^{-1}v)_{\lambda \in \Lambda},(\theta(\lambda)v)_{\lambda \in \Lambda}^d} \\ &= M_{(1),(\theta(\lambda)u)_{\lambda \in \Lambda}^d,(\theta(\lambda)(V^{-1})^*u)_{\lambda \in \Lambda}}\end{aligned}$$

using dual frames $(\theta(\lambda)u)_{\lambda \in \Lambda}^d$ and $(\theta(\lambda)v)_{\lambda \in \Lambda}^d$ with the θ -pseudo-coherent structure (for example, the canonical duals).

As one can see, the above formulas for V^{-1} involve V^{-1} in the representation. Note that there are other representations of V^{-1} which avoid the use of V^{-1} and can be based on any given θ -pseudo-coherent frame:

Proposition V.7: [13] Let $V : \mathcal{H} \rightarrow \mathcal{H}$ be an invertible θ -pseudo-coherent frame-type operator $M_{(1),(\theta(\lambda)v)_{\lambda \in \Lambda},(\theta(\lambda)u)_{\lambda \in \Lambda}}$ and let $(g_\lambda)_{\lambda \in \Lambda} = (\theta(\lambda)g)_{\lambda \in \Lambda}$ be any θ -pseudo-coherent frame for \mathcal{H} . Then V^{-1} can be written as the θ -pseudo-coherent frame-type operator $M_{(1),(g_\lambda)_{\lambda \in \Lambda},(\tilde{h}_\lambda)_{\lambda \in \Lambda}}$, where $h_\lambda = \theta(\lambda)Vg$, $\lambda \in \Lambda$.

As already mentioned, the class of Θ -pseudo-coherent frames includes Gabor frames and coherent frames. Thus, all the results mentioned in this subsection are valid for Gabor frames and coherent frames.

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