An FRI Model for Asymmetric Pulse Train and Characterization of Ventricular Hypertrophy Condition

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Abstract—We address the problem of modelling asymmetric pulses within the framework of finite-rate-of-innovation (FRI) signals. We use the fractional Hilbert transform to model asymmetry in the pulse shape. The annihilating filter is employed to estimate the locations of the pulses. The asymmetry of the pulses manifests as a complex phase factor in the Fourier-domain expression of the signal. We model a single QRS complex of an electrocardiogram (ECG) signal as a sum of a symmetric Gaussian and its Hilbert transform, which is antisymmetric. Since the Hilbert transform of a Gaussian pulse is not compactly supported, we truncate it using a Hamming window. We apply the proposed technique to three ECG datasets: ECG of healthy person having normal cardiac conditions, person with left ventricular hypertrophy, and person suffering from right ventricular hypertrophy. A comparison of the estimated amplitude and asymmetry parameters for the three datasets is shown, and a preliminary diagnostic tool for characterization of ventricular hypertrophy is proposed.

I. INTRODUCTION

Vetterli et al. [1] proposed a sampling and reconstruction technique for a class of signals that can be represented by a finite number of parameters, termed as finite-rate-of-innovation (FRI) signals. An example of an FRI signal is a stream of pulses,

\[ x(t) = \sum_{\ell=1}^{L} a_{\ell} g(t - t_{\ell}), \]

where the pulse shape \( g(t) \) is known a priori. The FRI problem is to find \( \{a_{\ell}, t_{\ell}\}_{\ell=1}^{L} \) given samples of \( x(t) \). In applications such as radio detection and ranging (RADAR) [2], [3] and ultrasound imaging [4], [5], the received signal can be modelled as an FRI signal, where \( g(t) \) represents the transmitted pulse. In ultrasound imaging, the parameters \( a_{\ell} \) and \( t_{\ell} \) denote the amplitude and time-delay, respectively, of the \( \ell \)th pulse reflected from a scatterer.

The received signal may contain a delayed and modified version of the transmitted pulse. In such scenarios, the received signal is given as

\[ x(t) = \sum_{\ell=1}^{L} A_{\theta_{\ell}} \{g\} (t - t_{\ell}), \]

where the modified signal \( A_{\theta_{\ell}} \{g\} (t) \) is related to the transmitted pulse \( g(t) \) by the parameters \( \theta_{\ell} \in \mathbb{C}^{M} \). Nagesh and Seelamantula [6] proposed a model for the changing asymmetry in ultrasound reflections. They assumed that the transmitted pulse is symmetric around zero and the received pulses have varying degrees of asymmetry. The asymmetric pulse is modelled as

\[ A_{\theta_{\ell}} \{g\} (t) = \theta_{1,\ell} g(t) + \theta_{2,\ell} \frac{\partial g(t)}{\partial t}. \]

Baechler et al. [7] proposed a technique, termed as variable-pulse-width finite-rate-of-innovation (VPW-FRI) to model electrocardiogram (ECG) signals within a single channel as,

\[ x(t) = \sum_{\ell=1}^{L} c_{\ell} \frac{a_{\ell}^2}{a_{\ell}^2 + (t - t_{\ell})^2} + d_{\ell} \frac{(t - t_{\ell})^2}{a_{\ell}^2 + (t - t_{\ell})^2}, \]

where \( c_{\ell} \) and \( d_{\ell} \) control the degree of asymmetry in the pulses and the parameters \( a_{\ell} \) and \( t_{\ell} \) denote the width and location of the pulses, respectively. Each ECG pulse is modelled by seven VPW pulses. Nair et al. [8] proposed a multichannel extension of the VPW-FRI method by using a common annihilator approach.

In general, an ECG signal is measured by twelve leads placed at different locations on the human body. Due to differences in electrical conductivity between the heart and ECG recording sites, the measured ECG pulses have different degrees of asymmetry. A given set of asymmetry parameters of the ECG measurements can be used to classify different heart conditions. Suppose \( \theta_{k} \) represents the degree of asymmetry of ECG pulse measured in the \( k \)th lead. The set of parameters \( \{\theta_{1}, \theta_{2}, \ldots, \theta_{12}\} \), can be used to distinguish a normal heart from an abnormal one. In particular, the set of asymmetry factors of the pulses measured by chest leads \( V_{1} \) to \( V_{6} \) is different for a healthy person and a person suffering from left or right ventricular hypertrophy. Traditionally, medical practitioners measure the ECG readings on a millimetre scale to detect and diagnose left or right ventricular hypertrophy. In this paper, we show that the asymmetry of the pulse can be used as a parameter to characterize left and right ventricular hypertrophy.

The QRS complex of ECG signal denotes the electrical activity in the ventricles of the heart. In Fig. 2, the ECG measurements of a normal person having healthy cardiac conditions is shown. We note that, across channels \( V_{1} \) to \( V_{6} \),
the asymmetry of the QRS pulse changes whereas, within a given channel, it remains constant. Hence, we propose the following model for the $k^{th}$ channel QRS complex:

$$x_k(t) = \sum_{\ell=1}^{L_k} a_{k,\ell} A_{\theta_k}(g(t - t_{k,\ell})),$$

where $L_k$ is the number of pulses in the $k^{th}$ channel. The amplitude and location of the $\ell^{th}$ pulse in the $k^{th}$ channel are given by $a_{k,\ell}$ and $t_{k,\ell}$, respectively, and the asymmetry of the $k^{th}$ channel is specified by $\theta_k$. In this paper, we develop a fractional Hilbert transform (FrHT) based model for asymmetric pulses and show its applicability to model the QRS complex across different channels. The degree of asymmetry is controlled by the FrHT phase parameter $\theta_k$. The parameters $a_{k,\ell}$, $t_{k,\ell}$ and asymmetry factor $\theta_k$ are estimated by the annihilating filter method [9]. We show that the estimated parameters can be used to characterise the type of ventricular hypertrophy.

II. ECG BACKGROUND

A typical pulse of an ECG signal is shown in Fig. 1. It consists of five pulses denoted as P wave, QRS complex, and T wave. The pulses signify electrical activity at different locations of the heart. The amplitude of the QRS complexes with respect to the baseline varies with degree of asymmetry of the QRS complex, which is caused due to different heart conditions. QV$_k$, RV$_k$, and SV$_k$ denote the amplitudes of the Q, R, and S waves in the $k^{th}$ channel with respect to the baseline.

A. Ventricular Hypertrophy

Ventricular hypertrophy is a medical term for thickening of the ventricular walls of the heart. It can be diagnosed using the ECG signal [10], [11] measured at leads V$_1$ to V$_6$, ultrasound, and computed tomography (CT). Traditionally, based on the amplitudes of the R and S waves, the condition of the ventricles has been classified as follows:

1) Normal condition, in which the V$_1$ lead contains a small R wave (less than 5 mm) and a large S wave (less than 7 mm) as shown in Fig. 2. The amplitude of the R wave increases from V$_1$ to V$_6$, and correspondingly, the amplitude of the S wave decreases. V$_3$ and V$_4$ leads have equal-amplitude R and S waves. The changing R and S components in each of the channels give rise to varying degrees of asymmetry across channels.

2) Left ventricular hypertrophy, in which the ECG of a person diagnosed as having left ventricular hypertrophy is shown in Fig. 3. The outputs of the ECG leads are similar to that of a normal heart but the amplitudes of the waves are higher than normal. A diagnosis of left ventricular hypertrophy is made if either: (i) SV$_1 > 25$ or RV$_6 > 25$ mm, or (ii) SV$_1$+RV$_6 > 35$ mm.

3) Right ventricular hypertrophy, in which the ECG of a person diagnosed as having right ventricular hypertrophy is shown in Fig. 4. Right ventricular hypertrophy is diagnosed if: i) either RV$_1$ or SV$_6 > 7$ mm, or ii) RV$_1$+SV$_6 > 10$ mm.

III. PROPOSED METHOD TO CHARACTERIZE VENTRICULAR CONDITION

We model the QRS complex of ECG measurements of different channels by $x_k(t)$ in (4). The parameters that distinguish each channel are the amplitudes and asymmetry factors of QRS pulses, which are modelled by FrHT. First, we briefly discuss Hilbert transform and FrHT and then show how the parameters are estimated by the proposed method.

A. Hilbert transform

The Hilbert transform of a function $h(t)$ is given as $\tilde{h}(t) = h(t) * \frac{1}{\pi t}$, where the convolution must be understood in the
Cauchy principal value sense. The definition of the Hilbert transform leads to the following proposition.

**Proposition 1**: The Hilbert transform of a pulse that is symmetric around zero, is anti-symmetric, that is, if \( h(t) = h(-t) \), then \( \tilde{h}(-t) = -\tilde{h}(t) \).

**Proof**: Let \( h(t) \) be a symmetric pulse. From the definition of the Hilbert transform \([12]\) and symmetry of \( h(t) \), \( \tilde{h}(-t) \) is given by

\[
\tilde{h}(-t) = \frac{-1}{\pi t} * h(t) = -\tilde{h}(t),
\]

which completes the proof.

### B. Fractional Hilbert transform

The Hilbert transform operator is essentially a \( \pi/2 \) phase shifter, mapping a cosinusoid to a sinusoid

\[
\cos(\omega_o t + \alpha) \xrightarrow{\mathcal{H}} \sin(\omega_o t + \alpha),
\]

where \( \mathcal{H} \) denotes the Hilbert transform operator. A generalisation of the Hilbert transform for a phase shift by an arbitrary angle \( \theta \) is the fractional Hilbert transform \([13], [14]\). The fractional Hilbert transform of a cosinusoid at an angle \( \theta \) is given by

\[
\cos(\omega_o t + \alpha) \xrightarrow{\mathcal{H}_\theta} \cos(\omega_o t + \alpha - \theta),
\]

where \( \mathcal{H}_\theta \) denotes the fractional Hilbert transform. The FrHT of \( h(t) \) denoted by \( h_\theta(t) \) is given by

\[
h_\theta(t) = \cos(\theta)h(t) + \sin(\theta)\tilde{h}(t).
\]

From Proposition 1, it is clear that if \( h(t) \) is symmetric, then \( \tilde{h}(t) \) is anti-symmetric, and any non-trivial linear combination of \( h(t) \) and \( \tilde{h}(t) \) is asymmetric. In Fig. 5(a), the symmetric pulse \( g(t) = e^{-t^2/2\sigma^2} \), with \( \sigma = 0.005 \) and its Hilbert transform \( \tilde{g}(t) \) are shown. The Hilbert transform of a Gaussian is computed by approximating \( \tilde{g}(t) \) with the discrete Hilbert transform of a finely sampled \( g(t) \). The discrete Hilbert transform is implemented by designing a finite-length Hilbert kernel \([12], [15]\). In Fig. 5(a) a symmetric Gaussian pulse and its Hilbert transform, which is anti-symmetric, are shown. The asymmetric pulses \( g_\theta(t) \) for \( \theta = 0.2\pi, 0.4\pi, 0.6\pi, \) and \( 0.8\pi \) are shown in Fig. 5(b).

### C. Asymmetric pulse modelling using FrHT

In this paper, we use the FrHT to relate \( g(t) \) to \( A_{\theta_k}(g)(t) \) in (4). In this model, \( \theta_k \in \mathbb{R} \) and the resulting model for the \( k \)-th channel QRS complex is given by

\[
x_k(t) = \sum_{\ell=1}^{L_k} a_{k,\ell} A_{\theta_k}(g)(t - t_{k,\ell}),
\]

\[
= \sum_{\ell=1}^{L_k} a_{k,\ell} (\cos(\theta_k)g(t - t_{k,\ell}) + \sin(\theta_k)\tilde{g}(t - t_{k,\ell})),
\]

where \( \{a_{k,\ell}, t_{k,\ell}\}_{\ell=1}^{L_k} \) are the amplitude, and time-location, respectively, of the \( \ell \)-th QRS pulse in the \( k \)-th channel. The asymmetry factor of the \( k \)-th channel is \( \theta_k \).

### D. Parameter estimation of asymmetric signals

The goal is to estimate the parameters \( \{a_{k,\ell}, \theta_k, t_{k,\ell}\}_{\ell=1}^{L_k} \) from \( x_k(t) \). Without loss of generality, assume that \( t_{k,1} < t_{k,2} < \cdots < t_{k,L_k} \) for \( k = 1, 2, \cdots, 6 \). The continuous-time Fourier transform (CTFT) of \( x_k(t) \) in (8) is given by

\[
X_k(\omega) = \sum_{\ell=1}^{L_k} a_{k,\ell} \cos(\theta_k) G(\omega)e^{-j\omega t_{k,\ell}} - j a_{k,\ell} \sin(\theta_k) G(\omega)e^{-j\omega t_{k,\ell}},
\]

\[
= \sum_{\ell=1}^{L_k} c_{k,\ell,\omega} G(\omega)e^{-j\omega t_{k,\ell}},
\]

where \( c_{k,\ell,\omega} = a_{k,\ell} \cos(\theta_k) - j a_{k,\ell} \sin(\theta_k) \). Define \( \mathcal{M}_k \) as a set of \( 2L_k \) consecutive positive integers, and \( \omega_o \in \mathbb{R}^+ \) such that \( G(m\omega_o) \neq 0 \), \( m \in \mathcal{M}_k \). We have

\[
X_k(m\omega_o) = G(m\omega_o) \sum_{\ell=1}^{L_k} c_{k,\ell,\omega} e^{-jm\omega_o t_{k,\ell}}, \quad \text{for } m \in \mathcal{M}_k,
\]

where \( c_{k,\ell,\omega} = a_{k,\ell}(\cos(\theta_k) - j \sin(\theta_k)) \). On choosing \( \omega_o \leq \frac{2\pi}{t_{k,L_k}} \), \( \{t_{k,\ell}\}_{\ell=1}^{L_k} \) can be uniquely estimated from the samples \( Y_k(m\omega_o) \triangleq \frac{X_k(m\omega_o)}{G(m\omega_o)} \), using spectral estimation techniques such as the annihilating filter \([16]\) or the matrix-pencil method \([17]\).

The complex amplitude \( c_{k,\ell} \) can be estimated from the frequency samples \( \hat{X}_k(m\omega_o) \) by solving the system of linear equations in (9). The amplitude and asymmetry factor are given by \( a_{k,\ell} = |c_{k,\ell}| \), and \( \theta_k = -\angle c_{k,\ell} \), for \( \ell \in [1, L_k] \), respectively. The phase \( \theta_k \) estimated from any of \( \{c_{k,\ell}\}_{\ell=1}^{L_k} \).
Fig. 7. (a) Reconstruction of signal when the Fourier samples of the signal are corrupted by noise (SNR = 50 dB), and (b) MSE of the reconstruction as function of \( \theta_k \).

Fig. 8. (a) Lead V1 output of the ECG signal of a normal person. (b) Gaussian pulse of variance 0.005 and its Hilbert transform. A Hamming window function used for truncation of the support of the Hilbert component of the signal. Also shown is the resulting truncated asymmetric pulse \( g_\theta(t) \), with \( \theta = 0.2\pi \).

will be the same. To validate the theory, we show reconstruction of an FRI signal of the form given in (8) for \( k = 1 \) with \( g(t) = e^{-t^2/2\sigma^2} \), \( \sigma = 0.009 \), and \( L_1 = 4 \), \( \{t_{1,\ell}\}_{\ell=1}^{L_1} = \{0.10, 0.3667, 0.6333, 0.9\} \), \( \{a_{1,\ell}\}_{\ell=1}^{L_1} = \{1, 2, 3, 4\} \), and \( \theta_1 = 0.2 \). The original and reconstructed signals are shown in Fig. 6 from where we observe that the reconstruction accuracy is high.

1) Parameter estimation in noise: Consider the Fourier samples \( Y_k(m\omega_o) \) corrupted by additive white Gaussian noise (AWGN) \( w[m] \),

\[
Z_k(m\omega_o) = \sum_{\ell=1}^{L_k} c_{k,\ell} e^{-jm\omega_o t_{k,\ell}} + w[m].
\]

The Cadzow algorithm [18] is used to denoise the frequency samples \( Z_k(m\omega_o) \). Subsequently, the annihilating filter technique is used to estimate the locations and amplitudes \( \{t_{k,\ell}, a_{k,\ell}\}_{\ell=1}^{L_k} \) from the denoised spectral samples. In the noiseless scenario, the phase can be estimated from any of the channels \( k = \{1, 2, \cdots, 6\} \). In the noisy scenario, we use the following approach to estimate \( \theta_k \).

\[
\arg \min_{\theta_k \in [0, 2\pi]} \| Z_k(m\omega_o) - \sum_{\ell=1}^{L_k} a_{k,\ell} e^{-jm\omega_o t_{k,\ell}} \|_2^2.
\]

In Fig. 7(a) we show the reconstruction of an asymmetric Gaussian signal in Fig. 6, when \( Y_k(m\omega_o) \) is corrupted by AWGN. The corresponding plot of the grid search for minimum mean-square-error (MSE) reconstruction is shown in Fig. 7(b).

Fig. 9. The detected QRS pulse shape in leads V1-V6 of the ECG recordings of a normal person using the Gaussian pulse shape.

**Algorithm 1 : ECG digitization**

1: Align the image \( I(m,n) \) using the Hough transform.
2: Remove gridlines on \( I(m,n) \) using hard-thresholding.
3: Remove salt and pepper noise by median filtering and binarizing the resulting image.
4: Rotate the image \( I(m,n) \) by 90\(^\circ\).
5: For row \( i \), compute the median of the distances of the dark pixels in row \( i \) from the left edge. This is the amplitude of the signal.
6: Repeat step 4 for every row of the image.
7: Interpolate the signal to obtain required sampling rate and apply the bilateral filter [19] to smooth the signal.

IV. ECG SIGNAL QRS DETECTION AND CHARACTERIZATION

We next apply the proposed FrHT technique for QRS detection of ECG signals and study the parameters obtained for various conditions of ventricular hypertrophy. The data used in this paper is taken from the ECG learning centre at the University of Utah [11]. The ECG signals shown in Figs. (3)-(5) are digitized using Algorithm 1 at a sampling frequency of 453 Hz. Fig. 8(a) shows the ECG signal in the V1 lead of a person having normal cardiac conditions. In the context of ECG modelling using asymmetric pulses, we experimentally found that the FrHT of Gaussian pulse \( g(t) = e^{-t^2/2\sigma^2} \) with \( \sigma = 0.005 \) accurately models the QRS complex of V1 to V6.
channels. The ECG signal in the \( k \)th channel is modelled as,
\[
y_k(t) = \sum_{\ell=1}^{L_k} a_{k,\ell} g_\ell(t - t_\ell) + w(t),
\]
where the QRS complex in the \( k \)th channel is given by \( \sum_{\ell=1}^{L_k} a_{k,\ell} g_\ell(t - t_\ell) \), and \( w(t) \) models the P and T pulses, and digitization noise. The Gaussian pulse \( g(t) \) and its Hilbert transform \( \tilde{g}(t) \) are shown in Fig. 8(b). Since the Hilbert transform of any pulse has infinite support, for practical purposes, \( \tilde{g}(t) \) should be truncated. We apply a Hamming window of length 0.07 s on \( \tilde{g}(t) \). Since it is the S wave that creates asymmetry in the QRS complex, the Hamming window is shifted to the right of the centre of the QRS complex. The resulting pulse shape \( g_\ell(t) \) for \( \theta = 0.2\pi \) is shown in Fig. 8(b).

The proposed FrHT based FRI algorithm is used to fit QRS pulses in the digitized ECG signals. The Cadzow denoising algorithm is employed to suppress noise present in the signal. The parameters \( \{a_{k,\ell}, L_k, \theta_k\} \) are obtained for the QRS pulses of leads V1 to V6. In Fig. 9, reconstructions of the QRS pulses of a person having normal cardiac conditions, for channels V1 to V6 are shown.

The FrHT based FRI algorithm is applied on the ECG datasets of normal cardiac conditions, right ventricular hypertrophy and left ventricular hypertrophy. The optimal \( \theta_k \) and an estimate of the average amplitude \( \hat{a}_k = \frac{1}{L_k} \sum_{\ell=1}^{L_k} a_{k,\ell} \) are given in Table I. We adopt a two-step procedure to classify a normal cardiac condition from left and right ventricular hypertrophy. First, an increasing sequence of \( \{\theta_k\}_{k=1}^{6} \) for the channels V1 to V6 indicates right ventricular Hypertrophy. Right ventricular hypertrophy is diagnosed by the presence of a large R wave in V1. This leads to a small value of \( \theta_k \). The optimal \( \theta_k \) for right ventricular hypertrophy increases from 0.6590 to 5.6370 from V1 to V6. A decreasing sequence of \( \{\theta_k\}_{k=1}^{6} \) indicates left ventricular Hypertrophy or a normal heart condition. The amplitudes of the channels can be used to classify between the two cardiac conditions. The presence of \( \{a_k\}_{k=1}^{6} \) higher than 2 in leads V1 to V6 is a good indication of a normal cardiac conditions and amplitudes \( \{a_k\}_{k=1}^{6} < 2 \) indicates left ventricular hypertrophy.

\[\text{normal condition from left and right ventricular hypertrophy conditions.}\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Leads} & \text{Normal} & \text{Left} & \text{Right} & \text{Normal} & \text{Left} & \text{Right} \\
\hline
V1 & 5.8550 & 5.7000 & 0.6590 & 1.0690 & 1.8332 & 0.4045 \\
V2 & 5.7270 & 5.2440 & 0.6080 & 2.1704 & 1.6742 & 0.6701 \\
V3 & 0.4936 & 0.7860 & 0.6650 & 2.1250 & 1.5401 & 1.0365 \\
V4 & 0.4300 & 0.5730 & 0.7760 & 1.8551 & 1.8744 & 1.4327 \\
V5 & 0.3840 & 0.4570 & 5.4390 & 1.6744 & 1.4327 & 0.7674 \\
V6 & 0.3230 & 0.2440 & 5.6370 & 1.4201 & 1.0712 & 0.6376 \\
\hline
\end{array}
\]

\section{V. Conclusions}

We have proposed a technique to model asymmetric pulses using the FrHT within the FRI framework. The fractional parameter of the Hilbert transform can be used to control the asymmetry in the pulses. Using the proposed technique, the QRS pulse shape is parameterized in terms of the amplitude and asymmetry in the pulse. Using the parameters of asymmetry, a diagnosis technique for ventricular hypertrophy is proposed. Rigorous validation of the proposed technique on more datasets is yet to be done.

\section{Acknowledgment}

The authors would like to thank Hari Sankar and Anicht Patni for developing the ECG digitization software.

\section{References}